Logic and Analysis (2008) 1: 153–185
DOI 10.1007/s11813-008-0006-0
Printed in The Netherlands

Logic and Analysis

Asymptotics of families of solutions of nonlinear difference equations

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Received 3 August 2007; Accepted 26 January 2008
Published online 9 May 2008;

Abstract One method to determine the asymptotics of particular solutions of a difference equation is by solving an associated asymptotic functional equation. Here we study the behaviour of the solutions in an asymptotic neighbourhood of such individual solutions. We identify several types of attraction and repulsion, which range from almost orthogonality to almost parallelness. Necessary and sufficient conditions for these types of behaviour are given.

Keywords Difference equations · asymptotics · stability · rivers · nonstandard analysis · change of scale

Mathematics Subject Classification (2000) 39A11 · 03H05

1 Introduction

Main results

We study first-order difference equations of the type

$$Y(X+1) = F(X, Y(X)), \tag{D}$$

where F is supposed to be continuously partially differentiable in Y. In [7] conditions were given in terms of F and F_2' to determine whether (D) possesses solutions with

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asymptotic behaviour $\hat{Y}(X)$. The sequence $\hat{Y}$ can be found by solving F for fixed points, i.e., $F(X, \hat{Y}(X)) = \hat{Y}(X)$, or whenever this is impossible or inconvenient for "almost fixed points" satisfying the so-called asymptotic functional equation

$$\lim_{X\to\infty} \frac{F(X, \hat{Y}(X)) - \hat{Y}(X)}{\hat{Y}(X)(|F_2'(X, \hat{Y}(X))| - 1)} = 0. \tag{A}$$

If such a weak, asymptotic fixed point for F satisfies some additional regularity conditions (see Definition 2.2) we speak of an *approximate solution*. The additional conditions express essentially that the solutions of (D) in a sufficiently wide neighbourhood of $\hat{Y}(X)$ contract (attract or repel each other), faster than $\hat{Y}(X)$ moves itself. An Existence Theorem, repeated here (Theorem 3.1), states that every approximate solution is asymptotic to a true solution $\tilde{Y}$ of (D). We used nonstandard analysis to prove this standard theorem.

In the present article we study in a precise asymptotic sense the stability of the solution $\tilde{Y}$. We suppose (D), $\hat{Y}$ and $\tilde{Y}$ to be standard and consider the "asymptotic halo" $H_{\tilde{Y}}$ of $\tilde{Y}$, given by

$$H_{\tilde{Y}} = \{(\omega, Y) \mid Y/\tilde{Y}(\omega) \simeq 1, \omega \simeq \infty\}.$$

The main theorems, Theorem 3.2, Theorem 3.3, and Theorem 3.4, present formulae for the deviation of solutions on $H_{\tilde{Y}}$, compared to the evolution of $\tilde{Y}$ itself. The formulae are descriptive and precise up to a multiplicative infinitesimal and we state necessary and sufficient conditions in terms of $F_2'(X, \hat{Y}(X))$ for them to hold on well-defined segments of $H_{\tilde{Y}}$. In a sense we complete the information on stability and deviation stemming from linearization through the variation equation $V(X+1) = F_2'(X, \hat{Y}(X))V(X)$ associated to (D) by asymptotic expressions for the solutions of the original equation when approaching or leaving the distinguished solution $\tilde{Y}$, and the domains where they are valid.

There appear to be five kinds of behaviour, linked to the order of magnitude of $F_2'(\omega, \hat{Y}(\omega))$, for unlimited ω. We illustrate this informally in the unstable or repulsive case, with $|F_2'(\omega, \hat{Y}(\omega))| > 1$; for precise information and exact conditions we refer to the theorems mentioned above. We let Φ be a second solution such that $\Phi(\omega)/\tilde{Y}(\omega) \simeq 1$.

If $F_2'(\omega, \hat{Y}(\omega)) \simeq \infty$, one has $(\Phi(\omega+1) - \tilde{Y}(\omega+1))/(\Phi(\omega) - \tilde{Y}(\omega)) \simeq \infty$ and $\tilde{Y}$ will be called a *strong river*. Observe that the repulsion is so strong that $\Phi(\omega+1)/\tilde{Y}(\omega+1)$ may be no longer infinitely close to 1. All other forms of repulsion will have the property that $\Phi(\omega+x)/\tilde{Y}(\omega+x)$ is infinitely close to 1 at least for all limited x.

If $(\Phi(\omega+x) - \tilde{Y}(\omega+x))/(\Phi(\omega) - \tilde{Y}(\omega)) \simeq a^x$ for some limited a with $|a| \gtrapprox 1$, the solution $\tilde{Y}$ will be called a *moderate river*, this corresponds to $F_2'(\omega, \hat{Y}(\omega))$ being limited, but not infinitely close to ± 1.

If $|F_2'(\omega, \hat{Y}(\omega))|$ is infinitely close to 1, it appears that we have to change scale, which turns the equation almost into a differential equation; its solutions are adequately described by (S-)continuous notions. Put $\gamma_\omega = |F_2'(\omega, \hat{Y}(\omega))| - 1$, which is infinitesimal, and $\delta(x) = (\Phi(\omega + x/\gamma_\omega) - \tilde{Y}(\omega + x/\gamma_\omega))/(\Phi(\omega) - \tilde{Y}(\omega))$. If γ_ω/ω is unlimited, one has $|\delta(x)| \simeq e^x$ for limited x and we call $\tilde{Y}$ a *weakly exponential river*. $|\delta|$ is not only S-continuous, it is in a sense nearly differentiable, for its difference quotient $(|\delta(x + \gamma_\omega)| - |\delta(x)|) / \gamma_\omega$ is also nearly equal to e^x. If $\gamma_\omega/\omega \equiv r$ is limited, it is equivalent to rescale at ω with γ_ω or with ω itself, for convenience we use the latter. If we denote $(\Phi(\omega + \omega x) - \tilde{Y}(\omega + \omega x))/((\Phi(\omega) - \tilde{Y}(\omega))$ again by $\delta(x)$, we derive that δ now is polynomial and satisfies $|\delta(x)| \simeq x^r$ for all limited x. Again $|\delta(x)|$ is nearly differentiable, with its difference quotient $(|\delta(x + 1/\omega)| - |\delta(x)|) / (1/\omega)$ nearly equal to the derivative rx^{r-1} of x^r. If $r > 0$, we call $\tilde{Y}$ a *polynomial river*. With this weak form of repulsion the secondary solution Φ still leaves the river $\tilde{Y}$, i.e., $\Phi(\xi)/\tilde{Y}(\xi)$ is no longer infinitely close to 1 for sufficiently large $\xi > \omega$. The weakest form of repulsion is observed for $r = 0$ in case $\sum_{X \geq C} \gamma_X$ is converging, where C is some natural number. Then $\Phi(\xi)/\tilde{Y}(\xi) \simeq 1$ for all $\xi \geq \omega$. To express this form of almost parallelness, $\tilde{Y}$ is called a *drain*.

In the case of stability or attraction one may make a similar distinction, with $|F_2'(\omega, \hat{Y}(\omega))| < 1$ decreasing from values infinitely close to 1 to infinitely close to 0.

Relation to existing literature

Our work is inspired by the so-called river phenomenon for differential equations. Computer graphics of the phase-portrait of several types of differential equations show contractions of trajectories, a striking optical phenomenon similar to rivers and its confluents on a map. The phenomenon is observed for such familiar equations as linear equations with constant coefficients, Riccati equations and the Van der Pol equation. Attempts to modelling were made in, among others, [12] (standard solutions acting as attractors or repellors of neighbouring solutions), [1] (slow solutions of slow-fast systems as attractors or repellors of neighbouring fast solutions), [4] (slow solutions of slow-fast systems as attractors or repellors of neighbouring fast solutions, after change of scale by macroscope), and [14, Chapter VII]. The latter presents a descriptive model in terms of exponential deviation from a central river solution, which is similar to the configurations described above. We observe that a large class of rivers satisfies an asymptotic functional equation similar to (A), i.e.,

$$\lim_{X \to \infty} \frac{F(X, \hat{Y}(X))}{\hat{Y}(X) F_2'(X, \hat{Y}(X))} = 0 \tag{R}$$

[3,4,14].

The class of equations considered in this article is the class of first-order difference equations (D). The equations may be nonautonomous and nonlinear and essentially only continuously partial derivability in order of Y is required. The class is larger than the classes which are usually studied, which are the class of linear equations [22–24], sometimes allowing for certain types of perturbations [2,13,20], and the class of analytic equations [16–18]. It is to be noted that these settings include equations of higher order and/or in more variables, which may be complex. The theories are more developed and notably the analytic theory gives more precision as to the asymptotics of the river solution. For instance, in the case of analytic equations one may look for a *formal solution* Y_0 in terms of a power series, for which one may show that it acts as the asymptotic expansion of an actual solution [16–18], and in the case of linear equations for expansions in terms of factorial series [22].

Within the limitations of first-order equations in one variable, the main theorems appear to be more general than the existing theorems on stability. Often they are stated in terms of eigenvalues in the case of linear homogeneous equations, sometimes allowing for certain types of generalizations [2,13]. These results are most close to our results on moderate rivers. The result on perturbations of linear equations presented in [13, Section 7.6] is perhaps the most general, because no conditions of regularity are imposed on the function F defining the equation. However only the autonomous case is considered and a strong condition is imposed on the boundedness of the perturbation. In our setting this would mean that $\sum_{X \geq A} |F_2'(X, \hat{Y}(X))|$ is bounded for some integer A, while we consider uncertainties of the form $o(\hat{Y}(X)(|F_2'(X, \hat{Y}(X))| - 1))$, which may be unbounded. The article [2] considers essentially perturbations of linear homogeneous equations and their nonzero eigenvalues, while in our setting $F_2'(X, \hat{Y}(X))$ may be asymptotically zero, or infinite. Still, we need that always $|F_2'(X, \hat{Y}(X))| < 1$, or always $|F_2'(X, \hat{Y}(X))| > 1$. In a sense this means that $\hat{Y}$ lies in an attractive tube, or in an repulsive tube. Also, our descriptive results, in terms of first-order asymptotic approximations of the solutions and the deviation of solutions, are more direct than the results using Lyapunov functions, like in [20].

We note some relation to the work [15] on singularly perturbed difference equations, because for a definite subclass of our equations there may be rescaled by macroscope to equations of this type.

The use of nonstandard analysis

Due to the distinction between standard and nonstandard, and internal and external properties, the expressive power of nonstandard analysis is stronger than that of classical analysis. Thus, behaviour may be modelled in a finer way, more close to the actual behaviour. We used the possibility to define models and state results in terms of external properties for the purpose of modelling a mathematical phenomenon which is necessarily approximate, i.e., the local evolution of a central solution and deviation from this central solution by a nearby family of other solutions. This choice also facilitates asymptotic calculations and reasoning. In the case of rivers of differential equations (see [14]) an attempt has been made to translate definitions and theorems into standard terms, where the natural setting seems to be the theory of regular variation [8]; this was possible at the price of introducing new parameters (see also [9]), epsilontics and long proofs based on the Transfer Principle.

The article is written in the axiomatic form *IST* of nonstandard analysis, which distinguishes itself from model-theoretic nonstandard analysis by the cohabitation of standard and nonstandard elements within infinite standard sets. Though a matter of taste, to our opinion for the purpose of modelling mathematical phenomena which are local and approximate, it is convenient to dispose of different orders of magnitudes within the set of real numbers.

For an introduction to *IST* and for terminology and notations we refer to [10,11,21]. We use external sets as in [19]. We denote by ∅ the external set of all infinitesimal numbers, £ the external set of all limited numbers, and @ the set of all positive appreciable (i.e., limited, but not infinitesimal) numbers. These symbols will be used just as $o(\cdot)$ and $O(\cdot)$ in classical asymptotics; for example, we may write $x = \varnothing$ instead of $x \simeq 0$ and several occurrences of ∅ in one formula may stand for different infinitesimals.

Structure of the article

In Section 2 we give formal definitions for approximate solutions and various kinds of local asymptotic behaviour of true solutions. Since the latter involve change of scale, we recall some of its methods, which are telescopes and macroscopes. The notions will be illustrated in terms of solvable, linear equations. In Section 3 we recall the existence theorem from [7] and present the three main theorems which characterize the local behaviour of families of solutions in terms of the partial derivative of the function defining the difference equation. In Section 4 we present a convenient lemma which states that though a nonlinear difference equation in general lacks the property of uniqueness of solutions, it is possible to move back in time for a subclass of its solutions. We prove also that in our setting repulsive solutions are unique in their asymptotic direction. Effects of the rescalings on the behaviour of the equation and its solutions are treated in Section 5. Section 6 contains the proofs of the main theorems. In Section 7 we consider some examples of quadratic equations (Section 7.1), a natural class of drains which admits a simple formula for deviations (Section 7.2), an obvious backward extension of the domains of the various types of behaviour through the uniqueness theorem of Section 4 (Section 7.3) and finally (Section 7.4) the question whether the river or drain itself satisfies the asymptotic functional equation (A).

2 Notations and definitions

Convention 2.1 Unless it is said explicitly to be otherwise, we always consider difference equations of the form $Y(X+1) = F(X, Y(X))$ (D), where F is a real-valued function which is defined and of class C^1 in the second variable on some set $U \subset N \times R$, such that the projection on N contains a set of the form $\{X \in \mathbb{N} \mid X \geq A_0\}$ with $A_0 \in \mathbb{N}$. We say that a sequence Y is a solution if $Y(X)$ is defined and satisfies (D) on some set $\{X \in \mathbb{N} \mid X \geq A_1\}$ with $A_1 \in \mathbb{N}$, or $Y(X)$ is defined on some set $\{X \in \mathbb{N} \mid A_2 \leq X \leq A_3\}$, with $A_2, A_3 \in \mathbb{N}$, $A_2 < A_3$ and satisfies (D) on $\{X \in \mathbb{N} \mid A_2 \leq X \leq A_3 - 1\}$; it is supposed that such an interval is maximal.

Definition 2.2 A sequence $\hat{Y}$ is called an *approximate solution* of (D) if

1. There exist $A \in \mathbb{N}$, $B \neq 0$ such that
   (a) Either $(\forall X \geq A)\,(\hat{Y}(X) < 0)$ or $(\forall X \geq A)\,(\hat{Y}(X) > 0)$.
   (b) $\{(X, Y) \mid X \geq A, (\exists \lambda)\,(0 \leq \lambda \leq 1, Y = \lambda(1-B)\hat{Y}(X) + (1-\lambda) \times (1+B)\hat{Y}(X))\} \subset U$.

(c) Either $(\forall X \geq A)\,(|F_2'(X, \hat{Y}(X))| < 1)$ or $(\forall X \geq A)\,(|F_2'(X, \hat{Y}(X))| > 1)$.

2. $\frac{F(X,\hat{Y}(X))-\hat{Y}(X)}{\hat{Y}(X)} = o(|F_2'(X, \hat{Y}(X))| - 1)$ for $X \to \infty$.
3. $\frac{\hat{Y}(X+1)-\hat{Y}(X)}{\hat{Y}(X)} = o(|F_2'(X, \hat{Y}(X))| - 1)$ for $X \to \infty$.
4. $Y(X) \sim \hat{Y}(X)$ for $X \to \infty$ implies $(|F_2'(X, Y(X))| - 1) \sim (|F_2'(X, \hat{Y}(X))| - 1)$ for $X \to \infty$.

Definition 2.3 Let $C \in \mathbb{N}$ be standard and $Z\colon \{C, \cdot\cdot, \infty\} \to \mathbb{R}$ be a standard nonzero sequence. We call

$$H_Z \equiv \{(\omega, Y) \mid \omega \simeq \infty, Y/Z(\omega) \simeq 1\}$$

the *asymptotic halo* of Z.

Assume (D) and $\hat{Y}$ to be standard. By Transfer U, A and B may be supposed standard. Then $H_{\hat{Y}} \subset U$. We will often use the following equivalent nonstandard form of the condition expressed in Definition 2.2.4:

$$(\forall \omega \simeq \infty)(\forall \eta \simeq 0)\left(\frac{|F_2'(\omega, (1+\eta)\hat{Y}(\omega))| - 1}{|F_2'(\omega, \hat{Y}(\omega))| - 1} \simeq 1\right). \tag{1}$$

The equivalence follows from Theorem 4.1(2) of [5]. Using nonstandard terminology, a geometric motivation of the conditions of Definition 2.2 has been given in [7], see also Section 7.4.

Definition 2.4 If $\hat{Y}$ is a standard approximate solution we put for all X such that $F_2'(X, \hat{Y}(X))$ is defined

$$g_X = |F_2'(X, \hat{Y}(X))| - 1, \quad \gamma_X = |g_X| = ||F_2'(X, \hat{Y}(X))| - 1|.$$

Definition 2.5 (See [8].) Let $H, K\colon \mathbb{N} \to \mathbb{R}$, where K is positive. We say that H is *slowly varying* at scale K if for $x \geq 0$ uniformly on every compact interval

$$\lim_{X \to \infty} \frac{H(X + xK(X))}{H(X)} = 1.$$

It follows from the nonstandard characterization of uniform convergence on compact intervals that, if H and K are standard, H is slowly varying at scale K if and only if for every unlimited ω and limited x

$$H(\omega + xK(\omega)) = (1 + \varnothing)H(\omega). \tag{2}$$

Note that for the definition making sense one must have $xK(\omega) \in \mathbb{N}$.

Definition 2.6 Let $\alpha \simeq 0$, $\alpha > 0$. Let $a, b \in \mathbb{R}$ be limited with $a \leq b$ and $(b-a)/\alpha \in \mathbb{N}$. We write

$$[a \cdot \cdot b] = \left\{ a + k\alpha \,\middle|\, k \in \mathbb{N}, 0 \leq k \leq \frac{b-a}{\alpha} \right\}.$$

The set $[a \cdot \cdot b]$ is called a *near-interval*. A function $f : [a \cdot \cdot b] \to \mathbb{R}$ is said to be *of class* S^0 on $[a \cdot \cdot b]$ if f is limited and S-continuous on $[a \cdot \cdot b]$. Put $\delta f(x) = f(x+\alpha) - f(x)$. The function f is said to be *of class* S^1 on $[a \cdot \cdot b]$ if f and $\delta f(x)/\alpha$ is of class S^0 on $[a \cdot \cdot b] \setminus \{b\}$. The function f is said to be *of class* $|S|^1$ on $[a \cdot \cdot b]$ if the function A_f defined by

$$A_f(x) = (-1)^{(x-a)/\alpha} f(x)$$

is of class S^1 on $[a \cdot \cdot b]$.

The shadow ${}^\circ\!f$ of a function f of class S^0 is a standard function of class C^0 [26]. The shadow ${}^\circ\!f$ of a function f of class S^1 is standard and of class C^1, and we have ${}^\circ\!\left(\frac{\delta f}{\alpha}\right)(x) = ({}^\circ\!f)'(x)$ for all $x \in [{}^\circ a, {}^\circ b]$ [10]; sometimes we write with abuse of language expressions like ${}^\circ\!\left(\frac{\delta f(x)}{\alpha}\right) = ({}^\circ\!f)'(x)$. Typically, alternating functions f such that $|f|$ is of class S^1 are of class $|S|^1$. It is easy to see that if f is of class $|S|^1$ the function $(f(x+\alpha) + f(x))/\alpha$ is of class S^0. Its shadow satisfies

$$ {}^\circ\!\left(\frac{f(x+\alpha) + f(x)}{\alpha}\right) = -\left({}^\circ\!A_f\right)'(x) \tag{3}$$

if $(x-a)/\alpha$ is even, and

$$ {}^\circ\!\left(\frac{f(x+\alpha) + f(x)}{\alpha}\right) = \left({}^\circ\!A_f\right)'(x) \tag{4}$$

if $(x-a)/\alpha$ is odd. In both cases the two-step difference quotient $(f(x+2\alpha) - f(x))/2\alpha$ is of class S^0 on $\left\{a + k \cdot 2\alpha \mid k \in \mathbb{N}, 0 \leq k < \frac{b-a}{2\alpha}\right\}$. Indeed, if f is of class S^1

$$\frac{f(x+2\alpha) - f(x)}{2\alpha} = \frac{1}{2}\frac{\delta f(x+\alpha)}{\alpha} + \frac{1}{2}\frac{\delta f(x)}{\alpha} \simeq \frac{\delta f(x)}{\alpha}, \tag{5}$$

and if f is of class $|S|^1$, one has for x such that $(x-a)/\alpha$ is even

$$\frac{f(x+2\alpha) - f(x)}{2\alpha} = \frac{1}{2}\frac{\delta A_f(x+\alpha)}{\alpha} + \frac{1}{2}\frac{\delta A_f(x)}{\alpha} \simeq \frac{\delta A_f(x)}{\alpha}, \tag{6}$$

and similarly for x such that $(x-a)/\alpha$ is odd,

$$\frac{f(x+2\alpha) - f(x)}{2\alpha} = -\frac{1}{2}\frac{\delta A_f(x+\alpha)}{\alpha} - \frac{1}{2}\frac{\delta A_f(x)}{\alpha} \simeq -\frac{\delta A_f(x)}{\alpha}. \tag{7}$$

We recall the general Stroboscopy Lemma of Callot and Sari [25,27] on the transition of difference equations to differential equations. Though essentially more general, when applied to near intervals $[a \cdot\cdot b]$ it says that the solution of a difference equation

$$\frac{\delta y(x)}{\alpha} = f(x, y), \quad y(a) \text{ limited,} \tag{8}$$

with f S-continuous on a standard neighbourhood of $(a, f(a))$ is of class S^1 at least up to some c with $a \lnapprox c \le b$; its shadow satisfies the differential equation

$$\frac{d({}^\circ y)(x)}{dx} = {}^\circ f(x, ({}^\circ y)(x)), \quad ({}^\circ y)({}^\circ a) = ({}^\circ y(a)). \tag{9}$$

at least on $[{}^\circ a, {}^\circ c]$.

However, sometimes one wishes more: the solution, say, η, of the initial value problem (9) should be defined on $[{}^\circ a, {}^\circ b]$ and be an infinitesimal approximation of the solution, say, θ, of (8) on $[a \cdot\cdot b] \cap [{}^\circ a, {}^\circ b]$. Then (9) should have existence and uniqueness of solutions for all $x \in [{}^\circ a, {}^\circ b]$. The equations considered in this article will always have this property. Indeed, the function ${}^\circ f$ will be of the form ${}^\circ f(x, y) = g(x)y$, with g defined and continuous on $[{}^\circ a, {}^\circ b]$. Then the solution η of (9) is defined and is unique on the whole of $[{}^\circ a, {}^\circ b]$. Also θ satisfies $\theta(x) \simeq \eta(x)$ for all $x \in [a \cdot\cdot b] \cap [{}^\circ a, {}^\circ b]$. The latter property may, for instance, be proved along the lines of the proof of the Strong Short-Shadow Lemma of [11]. This lemma is about infinitely closeness of a bounded solution ϕ of a standard differential equation and a solution ψ of an infinitely close nonstandard differential equation with an infinitely close initial condition, on the whole domain of definition of ϕ. Essential in the proof is that both equations have uniqueness of solutions. Note that if the nonstandard equation is a difference equation with infinitesimal increments, uniqueness of solutions is automatically satisfied.

We will consider two changes of scale for the set of solutions of difference equations, telescopes and macroscopes.

Definition 2.7 (*Telescope*) Let $\omega \in \mathbb{N}$ and $Z, L \in \mathbb{R}$ be such that $\omega \simeq \infty$, $Z \neq 0$ and $L > 0$, $L/\omega \simeq 0$. We define the telescope $T_{\omega,Z,L}$: $\mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{\omega,Z,L}(X, Y) = \left(\frac{X - \omega}{L}, \frac{Y - Z}{Z}\right).$$

Usually we write $x = (X - \omega)/(L)$ and $y = (Y - Z)/(Z)$. If Φ is a sequence, we write

$$\varphi_{\omega,Z,L}(x) = \frac{\Phi(\omega + Lx) - Z}{Z}.$$

In this notation, the difference equation (D) becomes

$$y_{\omega,Z,L}\left(x + \frac{1}{L}\right) = f_{\omega,Z,L}(x, y_{\omega,Z,L}(x)),$$

with

$$f_{\omega,Z,L}(x,y)=\frac{F(\omega+Lx,Z(1+y))}{Z}-1.$$

In this article we consider notably the cases $L=1$ and $L=1/\gamma_\omega$. Note that $\partial f_{\omega,Z,L}(x,y)/\partial y=\partial F(X,Y)/\partial Y$.

Definition 2.8 (*Macroscope*) Let $\omega\in\mathbb{N}$ and Z be such that $\omega\simeq\infty$ and $Z\neq 0$. We define the macroscope $M_{\omega,Z}$: $\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ by

$$M_{\omega,Z}(X,Y)=\left(\frac{X}{\omega},\frac{Y-Z}{Z}\right).$$

Usually we write $x=X/\omega$ and $y=(Y-Z)/Z$. If Φ is a sequence, we write

$$\varphi_{\omega,Z}(x)=\frac{\Phi(\omega x)-Z}{Z}.$$

In this notation, the difference equation (D) becomes

$$y_{\omega,Z}\left(x+\frac{1}{\omega}\right)=f_{\omega,Z}(x,y_\omega(x)),$$

with

$$f_{\omega,Z}(x,y)=\frac{F(\omega x,Z(1+y))}{Z}-1.$$

Note that again $\partial f_{\omega,Z}(x,y)/\partial y=\partial F(X,Y)/\partial Y$.

When there is no ambiguity with respect to the involved telescopes $T_{\omega,Z,L}$ we allow for the shorthand notation φ_ω for $\varphi_{\omega,Z,L}$ and f_ω for $f_{\omega,Z,L}$. Similarly, when there is no ambiguity with respect to the involved macroscopes $M_{\omega,Z}$ we allow also for the shorthand notation φ_ω for $\varphi_{\omega,Z}$ and f_ω for $f_{\omega,Z}$, noting that telescopes and macroscopes will be used in different settings.

In the context of standard difference equations it is necessary to consider a whole family of changes of scale: appropriate focussing depends in an essential way on the, possibly very individual, local behaviour at ω of the difference equation and the solution. So we use telescopes and macroscopes to rescale conveniently segments of the asymptotic halo of a standard nonzero solution $\tilde{Y}$. Indeed, let $\omega\simeq\infty$ and $L>0$ be such that $L/\omega\simeq 0$. Clearly, for the telescope $T_{\omega,\tilde{Y}(\omega),L}$ the external set $\{(x,y)\mid x$ limited, $y\simeq 0\}$ corresponds to $\{(X,Y)\mid X-\omega=\pounds L,\ Y/\tilde{Y}(\omega)\simeq 1\}$ and for the macroscope $M_{\omega,\tilde{Y}(\omega)}$ the external set $\{(x,y)\mid x$ appreciable, $y\simeq 0\}$ corresponds to $\{(X,Y)\mid X/\omega=@,\ Y/\tilde{Y}(\omega)\simeq 1\}$.

Using the rescalings mentioned above, the next definition describes several types of local asymptotic behaviour for sets of trajectories. For telescopes the description is made for positive limited x, but the described type of behaviour will be in fact verified for all limited x and for macroscopes the description is made for limited $x\geq 1$, while the described type of behaviour will be verified for all positive appreciable x; see Section 7.3.

Definition 2.9 Let (D) be standard and $\tilde{Y}$ be a nonzero solution. Let $\omega \in \mathbb{N}$.

1. Strong rivers. We call $\tilde{Y}$ a *strongly attractive river* if for every $\omega \simeq \infty$ under the telescope $T_{\omega,\tilde{Y}(\omega),1}$ for all limited $x \in \mathbb{N}$ it holds that ${}^\circ\tilde{y}_\omega(x) = 0$ and whenever Φ and Ψ are solutions with $\varphi_\omega(x) \simeq \psi_\omega(x) \simeq 0$, $\varphi_\omega(x) \neq \psi_\omega(x)$

$$\frac{\varphi_\omega(x+1) - \psi_\omega(x+1)}{\varphi_\omega(x) - \psi_\omega(x)} \simeq 0. \tag{10}$$

If instead of (10) it holds that $\varphi_\omega(0) \neq \psi_\omega(0)$ and for all limited $x \in \mathbb{N}$ such that $\varphi_\omega(\xi) \simeq \psi_\omega(\xi) \simeq 0$ for $\xi \in \{0, \cdot\cdot, x\}$

$$\frac{\varphi_\omega(x+1) - \psi_\omega(x+1)}{\varphi_\omega(x) - \psi_\omega(x)} \simeq +\infty$$

or for all such x

$$\frac{\varphi_\omega(x+1) - \psi_\omega(x+1)}{\varphi_\omega(x) - \psi_\omega(x)} \simeq -\infty,$$

we call $\tilde{Y}$ a *strongly repulsive river*.

2. Moderate rivers. We call $\tilde{Y}$ a *moderately attractive river* if for every $\omega \simeq \infty$ under the telescope $T_{\omega,\tilde{Y}(\omega),1}$ it holds that ${}^\circ\tilde{y}_\omega(x) = 0$ for all limited $x \geq 0$ and there exists standard a with $0 < |a| < 1$ such that whenever Φ and Ψ are solutions with $\varphi_\omega(0) \simeq \psi_\omega(0) \simeq 0$, $\varphi_\omega(0) \neq \psi_\omega(0)$ one has for all limited $x \in \mathbb{N}$

$$\frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(0) - \psi_\omega(0)} \simeq a^x. \tag{11}$$

If formula (11) holds for some standard a with $|a| > 1$, the solution $\tilde{Y}$ is called a *moderately repulsive river*.

3. Weakly exponential rivers. We call $\tilde{Y}$ a *weakly exponentially attractive river* if for every $\omega \simeq \infty$:

(a) $\gamma_\omega \simeq 0$, $\omega\gamma_\omega \simeq \infty$.
(b) Under the telescope $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ it holds that ${}^\circ\tilde{y}_\omega(x) = 0$ for all limited $x \geq 0$ and whenever Φ and Ψ are solutions with $\varphi_\omega(0) \simeq \psi_\omega(0) \simeq 0$, $\varphi_\omega(0) \neq \psi_\omega(0)$ one has for all limited $x \geq 0$

$$^\circ\left|\frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(0) - \psi_\omega(0)}\right| = e^{-x}. \tag{12}$$

Such a river is said to be *of class* S^1 if $\tilde{y}_\omega(x)$ and $(\varphi_\omega(x) - \psi_\omega(x))/(\varphi_\omega(0) - \psi_\omega(0))$ are of class S^1 and *of class* $|S|^1$ if $\tilde{y}_\omega(x) - \hat{y}_\omega(x)$ and $(\varphi_\omega(x) - \psi_\omega(x))/(\varphi_\omega(0) - \psi_\omega(0))$ are of class $|S|^1$. If the above properties hold with

$$^\circ\left|\frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(0) - \psi_\omega(0)}\right| = e^{x}. \tag{13}$$

the solution $\tilde{Y}$ is called a *weakly exponentially repulsive river (of class S^1 or $|S|^1$)*.

4. Polynomial rivers. We call $\tilde{Y}$ a *polynomially attractive river* if for every $\omega \simeq \infty$ under the macroscope $M_{\omega,\tilde{Y}(\omega)}$ it holds that ${}^{\circ}\tilde{y}_\omega(x) = 0$ for all limited $x \geq 1$ and there exists standard $r < 0$ such that whenever Φ and Ψ are solutions with $\varphi_\omega(1) \simeq \psi_\omega(1) \simeq 0$, $\varphi_\omega(1) \neq \psi_\omega(1)$ one has for all limited $x \geq 1$

$$^{\circ}\left|\frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(1) - \psi_\omega(1)}\right| = x^r. \tag{14}$$

Such a river is said to be *of class* S^1 if $\tilde{y}_\omega(x)$ and $(\varphi_\omega(x) - \psi_\omega(x))/(\varphi_\omega(1) - \psi_\omega(1))$ are of class S^1 and *of class* $|S|^1$ if $\tilde{y}_\omega(x) - \hat{y}_\omega(x)$ and $(\varphi_\omega(x) - \psi_\omega(x))/(\varphi_\omega(1) - \psi_\omega(1))$ are of class $|S|^1$. If the above properties hold with $r > 0$ the solution $\tilde{Y}$ is called a *polynomially repulsive river (of class* S^1 *or* $|S|^1$*).*

5. Drains. We call $\tilde{Y}$ a *drain* if for every $\omega \simeq \infty$ under the macroscope $M_{\omega,\tilde{Y}(\omega)}$ it holds that ${}^{\circ}\tilde{y}_\omega(x) = 0$ for all limited $x \geq 1$, and for every solution $\bar{Y}$ such that $\bar{Y}(\omega)/\tilde{Y}(\omega) \simeq 1$ it holds that $\bar{Y}(X)/\tilde{Y}(X) \simeq 1$ for all $X \simeq \infty$. The drain is said to be *of class* S^1 if for every $\omega \simeq \infty$ the discrete function $\tilde{y}_\omega(x)$ is of class S^1, and whenever Φ and Ψ are solutions with $\varphi_\omega(1) \simeq \psi_\omega(1) \simeq 0$, $\varphi_\omega(1) \neq \psi_\omega(1)$, the discrete function $(\varphi_\omega(x) - \psi_\omega(x))/(\varphi_\omega(1) - \psi_\omega(1))$ is of class S^1. The drain is said to be *of class* $|S|^1$ if for every $\omega \simeq \infty$ the discrete function $\tilde{y}_\omega(x) - \hat{y}_\omega(x)$ is of class $|S|^1$.

Remarks. 1. The "central solution" $\tilde{Y}$ in Definition 2.9 is always supposed to be standard. This is in line with the observation in [12], that solutions with standard initial conditions may act as remarkable solutions, solutions of reference in the phase-portrait. In our case such solutions are standard solutions, because (D) is standard, and act as attractors or repellors.

2. In order to be of class $|S|^1$ a discrete function needs to be "smoothly oscillating". It appears that $\tilde{y}_\omega$ is "smoothly oscillating" around $\hat{y}_\omega$ (Definitions 2.9.3, 2.9.4 and 2.9.5) under the conditions of Theorems 3.2.4, 3.2.6, 3.3.4, 3.3.6 and 3.4.2, where $\hat{y}_\omega$ itself is of class S^1 (Lemmas 5.2 and 5.5).

3. It is interesting to observe the transition of moderate rivers and polynomial rivers into weakly exponential rivers as a function of $F_2'(\omega, \hat{Y}(\omega))$. Assume that in Definition 2.9.2 we had viewed the difference of two solutions under the telescope $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ instead of under the telescope $T_{\omega,\tilde{Y}(\omega),1}$. Then we would have found

$$^{\circ}\left|\frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(0) - \psi_\omega(0)}\right| = (1 + g_\omega)^{x/\gamma_\omega}. \tag{15}$$

By Euler's formula, as γ_ω is approaching infinitesimal values, formula (15) turns into formula (12) or formula (13) of Definition 2.9.3, depending on whether $|F_2'(\omega, \hat{Y}(\omega))| < 1$ or $|F_2'(\omega, \hat{Y}(\omega))| > 1$.

Also, if in Definition 2.9.4 we had viewed the difference of two solutions under the change of scale $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ (which is formally not a telescope, since $\gamma_\omega = (1 + \oslash)r/\omega$), instead of under the macroscope $M_{\omega,\tilde{Y}(\omega)}$, we would have found

$$^{\circ}\left|\frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(0) - \psi_\omega(0)}\right| = \left(1 + \frac{x}{|r|}\right)^r \tag{16}$$

for $x \geq 0$.

Again by Euler's formula, as r is approaching negative or positive unlimited values, formula (16) turns into formula (12) or formula (13).

Examples. Using linear equations we give an example of each type of behaviour. The parameter a is always supposed to be standard.

1. *Strong rivers*. Consider

$$Y(X+1) = X^a Y(X) - X^{a+2} + X^2 + 2X + 1.$$

The solution $\tilde{Y}(X) = X^2$ is a strongly attractive river if $a < 0$ and a strongly repulsive river if $a > 0$.

2. *Moderate rivers*. Consider

$$Y(X+1) = aY(X) - X + \frac{1}{a-1}.$$

The solution $\tilde{Y}(X) = X/(a-1)$ is a moderately attractive river if $0 < |a| < 1$ and a moderately repulsive river if $|a| > 1$.

3. *Weakly exponential, polynomial rivers, drains of class* S^1. Consider

$$Y(X+1) = \left(1 - \frac{1}{X^a}\right) Y(X) + \frac{1}{X^a}.$$

The solution $\tilde{Y}(X) = 1$ is a weakly exponentially attractive river of class S^1 if $0 < a < 1$ and a polynomially attractive river of class S^1 if $a = 1$. It is a drain of class S^1 if $a > 1$. As for the repulsive case, consider

$$Y(X+1) = \left(1 + \frac{1}{X^a}\right) Y(X) - \frac{1}{X^a}.$$

The solution $\tilde{Y}(X) = 1$ is a weakly exponentially repulsive river of class S^1 if $0 < a < 1$ and a polynomially repulsive river of class S^1 if $a = 1$. It is a drain of class S^1 if $a > 1$. It will be shown that the solutions in the asymptotic halo of $\tilde{Y}$ satisfy the more precise formulae (43) and (44).

4. *Weakly exponential, polynomial rivers, drains of class* $|S|^1$. Consider

$$Y(X+1) = \left(-1 + \frac{1}{X^a}\right) Y(X) + 2 - \frac{1}{X^a} + (-1)^X \left(\frac{1}{X^a} - \frac{1}{(X+1)^a} - \frac{1}{X^{2a}}\right). \tag{17}$$

An approximate solution is given by $\hat{Y}(X) = 1$. The solution $\tilde{Y}(X) = 1 + (-1)^X/X^a$ is a weakly exponentially attractive river of class $|S|^1$, but not of class S^1 if $0 < a < 1$ and a polynomially attractive river of class $|S|^1$, but not of class S^1 if $a = 1$. It is a drain of class $|S|^1$, but not of class S^1 if $a > 1$. As for the repulsive case, consider

$$Y(X+1) = \left(-1 - \frac{1}{X^a}\right) Y(X) + 2 + \frac{1}{X^a} + (-1)^X \left(\frac{1}{X^a} - \frac{1}{(X+1)^a} + \frac{1}{X^{2a}}\right).$$

Again $\hat{Y}(X) = 1$ is an approximate solution and $\tilde{Y}(X) = 1 + (-1)^X/X^a$ is a solution. The latter is a weakly exponentially repulsive river of class $|S|^1$, but not of class S^1 if $0 < a < 1$ and a polynomially repulsive river of class $|S|^1$, but not of class S^1 if $a = 1$. It is a drain of class $|S|^1$, but not of class S^1 if $a > 1$. Then the solutions in the asymptotic halo of $\tilde{Y}$ satisfy also the finer formulae (43) and (44).

In Section 7.1 we consider more examples, then of nonlinear equations.

3 Existence and characterization theorems

The theorems below enable to decide from the properties of the function F defining the difference equation (D) whether in a given asymptotic direction it admits a family of solutions exhibiting one of the types of behaviour described in Section 2.

We start by recalling the Existence Theorem of [7], which states that the regularity conditions of Definition 2.2.1, 2.2.3 and 2.2.4 are sufficient for an approximate solution $\hat{Y}$, found by solving the asymptotic functional equation (A) of definition 2.2.2 to be asymptotic to a true solution.

Theorem 3.1 (Existence Theorem) *Let $\hat{Y}$ be an approximate solution of (D). Then (D) has a solution $\tilde{Y}$ such that $\tilde{Y}(X) \sim \hat{Y}(X)$ for $X \to \infty$.*

By transfer, if (D) and $\hat{Y}$ are standard, there exists a standard solution $\tilde{Y}$ such that $\tilde{Y}(X) \sim \hat{Y}(X)$ for $X \to \infty$. Then $\tilde{Y}(\omega)/\hat{Y}(\omega) \simeq 1$ for all $\omega \simeq \infty$. The next theorems characterize the behaviour of the set of solutions of (D) on the asymptotic halo of $\tilde{Y}$ (or equivalently of $\hat{Y}$) in terms of $F_2'(\omega, \hat{Y}(\omega))$ (or equivalently of $F_2'(\omega, \tilde{Y}(\omega))$). The theorems will be proved in Sections 4–6.

Theorem 3.2 (Characterization Theorem for attractive rivers) *Let (D) be a standard difference equation and $\hat{Y}$ be a standard approximate solution. Let $\tilde{Y}$ be a standard solution asymptotic to $\hat{Y}$.*

1. *The solution $\tilde{Y}$ is a strongly attractive river if and only if*
   (a) $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 0.$
   (b) $\lim_{X\to\infty} \hat{Y}(X+1)/\hat{Y}(X) = 1.$
2. *The solution $\tilde{Y}$ is a moderately attractive river if and only if*
   (a) $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = a$ *for some a with* $0 < |a| < 1.$
   (b) $\lim_{X\to\infty} \hat{Y}(X+1)/\hat{Y}(X) = 1.$
3. *The solution $\tilde{Y}$ is a weakly exponentially attractive river of class S^1 if and only if*
   (a) $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 1.$
   (b) $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) - 1) = -\infty.$
   (c) $F_2'(X, \hat{Y}(X)) - 1$ *is slowly varying at scale* $1/||F_2'(X, \hat{Y}(X))| - 1|.$
4. *The solution $\tilde{Y}$ is a weakly exponentially attractive river of class $|S|^1$, but not of class S^1 if and only if*
   (a) $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = -1.$
   (b) $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) + 1) = +\infty.$
   (c) $F_2'(X, \hat{Y}(X)) + 1$ *is slowly varying at scale* $1/||F_2'(X, \hat{Y}(X))| - 1|.$

5. *The solution* $\tilde{Y}$ *is a polynomially attractive river of class* S^1 *if and only if* $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X)) - 1) = r$ *for some* $r < 0$.
6. *The solution* $\tilde{Y}$ *is a polynomially attractive river of class* $|S|^1$, *but not of class* S^1 *if and only if* $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X)) + 1) = r$ *for some* $r > 0$.

Theorem 3.3 (Characterization Theorem for repulsive rivers) *Let* (D) *be a standard difference equation and* $\hat{Y}$ *be a standard approximate solution. Let* $\tilde{Y}$ *be a standard solution asymptotic to* $\hat{Y}$. *Repulsive rivers of the various types are characterized by the conditions of Theorem* 3.2 *with the following modifications*:

1. *Strongly repulsive rivers – instead of* 1a,

$$\lim_{X\to\infty} F_2'(X,\hat{Y}(X)) = +\infty \text{ or } \lim_{X\to\infty} F_2'(X,\hat{Y}(X)) = -\infty.$$

2. *Moderately repulsive rivers – instead of* $0 < |a| < 1$ *in* 2a, $|a| > 1$.
3. *Weakly exponentially repulsive rivers of class* S^1 *– instead of* 3b,

$$\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X)) - 1) = +\infty.$$

4. *Weakly exponentially repulsive rivers of class* $|S|^1$, *but not of class* S^1 *– instead of* 4b,

$$\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X)) + 1) = -\infty.$$

5. *Polynomially repulsive rivers of class* S^1 *– instead of* $r < 0$ *in* 5, $r > 0$.
6. *Polynomially repulsive rivers of class* $|S|^1$, *but not of class* S^1 *– instead of* $r > 0$ *in* 6, $r < 0$.

Theorem 3.4 (Characterization Theorem for drains) *Let* (D) *be a standard difference equation and* $\hat{Y}$ *be a standard approximate solution. Let* $\tilde{Y}$ *be a standard solution asymptotic to* $\hat{Y}$.

1. *The solution* $\tilde{Y}$ *is a drain of class* S^1 *if and only if*
   (*a*) $\sum_{X\geq C} F_2'(X,\hat{Y}(X)) - 1$ *is converging, for some* $C > 0$.
   (*b*) $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X)) - 1) = 0$.
2. *The solution* $\tilde{Y}$ *is a drain of class* $|S|^1$, *but not of class* S^1 *if and only if*
   (a) $\sum_{X\geq C} F_2'(X,\hat{Y}(X)) + 1$ *is converging, for some* $C > 0$.
   (b) $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X)) + 1) = 0$.

4 On the uniqueness of solutions

By no means nonlinear difference equations satisfy the property of uniqueness of solutions, as is immediately clear by considering quadratic equations. However, uniqueness may be satisfied locally. We will consider two cases for equations (D) which admit an approximate solution $\hat{Y}$. First we show that uniqueness is satisfied if we constrain ourselves to solutions within an appropriate tube around $\hat{Y}$. Secondly, we consider solutions asymptotic to $\hat{Y}$ which exist by the Existence Theorem. Clearly, in the attractive case there are an infinity of such solutions. But we show that in the repulsive case there exists only one.

Next definition and proposition are stated for general difference equations.

Definition 4.1 Let (D) be a difference equation defined on $U \subset \mathbb{N}\times\mathbb{R}$. Let $V \subset U$. We say that (D) satisfies the *property of uniqueness of solutions on* V if for all solutions Φ and Ψ such that for all X_0, X_1 with $X_0 \leq X_1$ it holds that if $\Phi_{|\{X_0,..,X_1\}}, \Psi_{|\{X_0,..,X_1\}} \subset V$

$$\Phi(X_1) = \Psi(X_1) \Longrightarrow \Phi_{|\{X_0,..,X_1\}} = \Psi_{|\{X_0,..,X_1\}}.$$

Note that if $\Phi(X_1) = \Psi(X_1)$, automatically $\Phi(X) = \Psi(X)$ for $X \geq X_1$ as long as $\Phi(X)$ and $\Psi(X)$ are both defined. We allow for nonuniqueness outside V, i.e., it may happen that for some solution Θ and $X > X_0$ one has $\Phi_{|\{X,..,X_1\}} = \Theta_{|\{X,..,X_1\}}$, but $\Phi(X-1) \neq \Theta(X-1)$, with $(X-1, \Theta(X-1)) \notin V$.

Proposition 4.2 *Let $Y(X+1) = F(X, Y(X))$ (D) be a difference equation defined on $U \subset \mathbb{N}\times\mathbb{R}$. Let $V \subset U$. Assume for all $X \in \mathbb{N}$ such that there exists some $Y \in \mathbb{R}$ with $(X, Y) \in V$ the function $F(X, .)$ is injective on $V \cap \{X\}\times\mathbb{R}$. Then (D) satisfies the property of uniqueness of solutions on V.*

Proof Let Φ and Ψ be solutions and $X_0, X_1 \in \mathbb{N}$ with $X_0 \leq X_1$, $\Phi_{|\{X_0,..,X_1\}}$, $\Psi_{|\{X_0,..,X_1\}} \subset V$ and $\Phi(X_1) = \Psi(X_1)$. We prove that $\Phi_{|\{X_0,..,X_1\}} = \Psi_{|\{X_0,..,X_1\}}$ by downward induction. Let $X_0 < X \leq X_1$ and assume $\Phi(X) = \Psi(X)$. Now $(X-1, \Phi(X-1)) \in V$ and $(X-1, \Psi(X-1)) \in V$, while $F(X-1, \Phi(X-1)) = \Phi(X) = \Psi(X) = F(X-1, \Psi(X-1))$. Because $F(X, .)$ is injective on $V \cap \{X\}\times\mathbb{R}$, one has $\Phi(X-1) = \Psi(X-1)$. □

We apply Proposition 4.2 to an appropriate tube around an approximate solution.

Lemma 4.3 (Uniqueness Lemma) *Let (D) be a standard difference equation satisfying the conditions of convention 2.1 and $\hat{Y}$ be a standard approximate solution. Assume* $\liminf_{X\to\infty} |F_2'(X, \hat{Y}(X))| \neq 0$. *Then there exists standard A_0, B_0 with $B_0 > 0$ such that (D) satisfies the property of uniqueness on $V \equiv \{(X, Y) \mid X \geq A_0, (\exists\lambda)\,(0 \leq \lambda \leq 1, Y = \lambda(1-B_0)\hat{Y}(X) + (1-\lambda)(1+B_0)\hat{Y}(X))\}$.*

Proof We have $|F_2'(X, \hat{Y}(X))| \not\simeq 0$ for all $X \simeq \infty$. Then $F_2'(X, Y) \not\simeq 0$ for all $X \simeq \infty$ and Y such that $Y/\hat{Y}(X) \simeq 1$ by formula (1). By the Fehrele principle [11], applied in two dimensions we obtain an internal set $I \supset H_{\hat{Y}}$ such that $F_2'(X, Y) \not\simeq 0$ on I. Such a set I contains a "rectangle" of type V for some standard A_0, B_0 with $B_0 > 0$. Then $F(X, Y)$ is injective in the second variable as long as $(X, Y) \in V$. Hence (D) satisfies the property of uniqueness on V. □

The next theorem states that within a tube V as given in the Uniqueness Lemma, a difference equation (D) has only one solution asymptotic to a repulsive approximate solution $\hat{Y}$.

Theorem 4.4 (Uniqueness Theorem for the repulsive case). *Let $\hat{Y}$ be an approximate solution of (D). Suppose there exists $A \in \mathbb{N}$ such that $(\forall X \geq A)\ (|F_2'(X, \hat{Y}(X))| > 1)$. Let $A_0 \in \mathbb{N}, A_0 \geq A, B_0 > 0$ such that (D) satisfies the property of uniqueness on $V \equiv \{(X, Y) \mid X \geq A_0, (\exists\lambda)\ (0 \leq \lambda \leq 1, Y = \lambda(1 - B_0)\hat{Y}(X) + (1 - \lambda)(1 + B_0)\hat{Y}(X))\}$. Consider the restriction of (D) to V. Within this restriction there exists a unique solution $\tilde{Y}$ such that $\tilde{Y}(X) \sim \hat{Y}(X)$ for $X \to \infty$.*

Proof By Transfer, we may suppose (D), A, $\hat{Y}$ and V to be standard. By the Existence Theorem there is a solution $\tilde{Y}$ such that $\tilde{Y}(X) \sim \hat{Y}(X)$ for $X \to \infty$. Let $A_1 \geq A_0$ be minimal such that $(X, \tilde{Y}(X)) \in V$ for all $X \geq A_1$. Suppose $\bar{Y}$ is a (standard) different solution within V, also asymptotic to $\hat{Y}$. By convention 2.1 the solutions $\bar{Y}$ and $\tilde{Y}$ are maximal, so $\bar{Y}(X) \neq \tilde{Y}(X)$ for all X such that they are both defined within V. In particular $\bar{Y}(X) \neq \tilde{Y}(X)$ for all $X \simeq \infty$. Let $\omega \simeq \infty$. Because $\lim_{X\to\infty}(\bar{Y}(X) - \tilde{Y}(X))/\hat{Y}(X) = 0$, there exists $\xi > \omega$ such that $|(\bar{Y}(\xi) - \tilde{Y}(\xi))/\hat{Y}(\xi)| < |(\bar{Y}(\omega) - \tilde{Y}(\omega))/\hat{Y}(\omega)|$. Hence for some X with $\omega \leq X \leq \xi$ one has $|(\bar{Y}(X + 1) - \tilde{Y}(X + 1))/\hat{Y}(X + 1)| < |(\bar{Y}(X) - \tilde{Y}(X))/\hat{Y}(X)|$. So

$$\left|\frac{F(X, \bar{Y}(X)) - F(X, \tilde{Y}(X))}{\bar{Y}(X) - \tilde{Y}(X)}\right| - 1 < \left|\frac{\hat{Y}(X + 1)}{\hat{Y}(X)}\right| - 1. \tag{18}$$

By the Mean Value Theorem and formula (1) there exists $\alpha \simeq 0$ such that the left-hand side of (18) is equal to $(1 + \alpha)g_X$, and by Definition 2.2.3 there exists $\beta \simeq 0$ such that the right-hand side of (18) is equal to $-1 + |1 + \beta g_X|$. So formula (18) becomes

$$\begin{cases} (1 + \alpha - \beta)g_X < 0 & 1 + \beta g_X \geq 0 \\ (1 + \alpha + \beta)g_X < -2 & 1 + \beta g_X < 0. \end{cases}$$

In both cases we conclude that $g_X < 0$, i.e., $|F_2'(X, \hat{Y}(X))| < 1$, a contradiction. Hence $\tilde{Y}$ is the only solution asymptotic to $\hat{Y}$. □

5 Rescalings

We derive some approximation lemmas which yield useful information on the behaviour of the difference equation (D) viewed under the telescopes or macroscopes. All lemmas are formulated for the case where (D), the approximate solution $\hat{Y}$ and the solution $\tilde{Y}$ asymptotic to $\hat{Y}$ are standard, and ω is a nonstandard integer. The following approximations hold for $\hat{Y}$, $\tilde{Y}$ and F_2'.

Lemma 5.1 *Let* $\lim_{X\to\infty} \hat{Y}(X + 1)/\hat{Y}(X) = 1$. *Then* ${}^\circ\hat{y}_\omega = {}^\circ\tilde{y}_\omega = 0$ *under the telescope* $T_{\omega,\tilde{Y}(\omega),1}$.

We omit the proof.

Lemma 5.2 *Let* $\lim_{X\to\infty} |F_2'(X, \hat{Y}(X))| = 1$, $\lim_{X\to\infty} X \cdot ||F_2'(X, \hat{Y}(X))| - 1| = \infty$ *and* $|F_2'(X, \hat{Y}(X))| - 1$ *be slowly varying at scale* $1/||F_2'(X, \hat{Y}(X))| - 1|$. *Then* ${}^\circ\hat{y}_\omega = {}^\circ\tilde{y}_\omega = 0$ *under the telescope* $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$. *Moreover,* $\hat{y}_\omega$ *is of class* S^1.

Proof Let $x \geq 0$ be limited. We have $\gamma_X/\gamma_\omega \simeq 1$ for all X with $\omega \leq X < \omega + x/\gamma_\omega$. Applying Definition 2.2.3 and Euler's formula

$$\frac{\hat{Y}(\omega + x/\gamma_\omega)}{\hat{Y}(\omega)} = \prod_{\omega \leq X < \omega + x/\gamma_\omega} 1 + \frac{\hat{Y}(X+1) - \hat{Y}(X)}{\hat{Y}(X)}$$
$$= \prod_{\omega \leq X < \omega + x/\gamma_\omega} 1 + \oslash\gamma_X = (1 + \oslash\gamma_\omega)^{x/\gamma_\omega} = \exp(\oslash x) = 1 + \oslash.$$

Also
$$\frac{\tilde{Y}(\omega + x/\gamma_\omega)}{\tilde{Y}(\omega)} = \frac{\tilde{Y}(\omega + x/\gamma_\omega)}{\hat{Y}(\omega + x/\gamma_\omega)} \frac{\hat{Y}(\omega + x/\gamma_\omega)}{\hat{Y}(\omega)} \frac{\hat{Y}(\omega)}{\tilde{Y}(\omega)} \simeq 1.$$

So ${}^\circ\tilde{y}_\omega(x) = 0$, hence also ${}^\circ\hat{y}_\omega(x) = 0$. Further

$$\frac{\hat{y}_\omega(x + \gamma_\omega) - \hat{y}_\omega(x)}{\gamma_\omega}$$
$$= \frac{\hat{Y}(\omega + x/\gamma_\omega + 1) - \hat{Y}(\omega + x/\gamma_\omega)}{\hat{Y}(\omega + x/\gamma_\omega)\,\gamma_{\omega + x/\gamma_\omega}} \frac{\gamma_{\omega + x/\gamma_\omega}}{\gamma_\omega}(1 + \hat{y}_\omega(x)) = \oslash(1+\oslash)(1+\oslash) = \oslash.$$

Hence $\hat{y}_\omega$ is of class S^1 for all limited $x \geq 0$. Clearly the above formulae hold also for negative limited x. □

Lemma 5.3 *Let* $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 1$, $\lim_{X\to\infty} X \cdot ||F_2'(X, \hat{Y}(X))| - 1| = \infty$ *and* $F_2'(X, \hat{Y}(X)) - 1$ *be slowly varying at scale* $1/||F_2'(X, \hat{Y}(X))| - 1|$. *Then under the telescope* $T_{\omega, \tilde{Y}(\omega), 1/\gamma_\omega}$ *one has* $(\partial f_\omega/\partial y)(x, y) = 1 + (1 + \oslash)g_\omega$ *for all limited* x *and* $y \simeq 0$.

Proof We have

$$\frac{\partial F}{\partial Y}(\omega + x/\gamma_\omega, (1+y)\tilde{Y}(\omega)) = 1 + \frac{\frac{\partial F}{\partial Y}(\omega + x/\gamma_\omega, (1+y)\tilde{Y}(\omega)) - 1}{\frac{\partial F}{\partial Y}(\omega + x/\gamma_\omega, \hat{Y}(\omega + x/\gamma_\omega)) - 1} \times$$
$$\times \frac{\frac{\partial F}{\partial Y}(\omega + x/\gamma_\omega, \hat{Y}(\omega + x/\gamma_\omega)) - 1}{\frac{\partial F}{\partial Y}(\omega, \hat{Y}(\omega)) - 1} \cdot \left(\frac{\partial F}{\partial Y}(\omega, \hat{Y}(\omega)) - 1\right).$$

Applying Lemma 5.2, formula (1) and the slow variation of $F_2'(X, \hat{Y}(X)) - 1$ at scale $1/||F_2'(X, \hat{Y}(X))| - 1|$ we obtain for all limited x and $y \simeq 0$ that

$$\frac{\partial f_\omega}{\partial y}(x, y) = 1 + (1+\oslash)(1+\oslash)g_\omega = 1 + (1+\oslash)g_\omega.$$

□

Lemma 5.4 *Let* $\lim_{X\to\infty} F_2'(X,\hat{Y}(X)) = -1$, $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X))+1) = +\infty$ *and* $F_2'(X,\hat{Y}(X))+1$ *be slowly varying at scale* $1/||F_2'(X,\hat{Y}(X))|-1|$. *Then under the telescope* $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ *one has* $(\partial f_\omega/\partial y)(x,y) = -1-(1+\oslash)g_\omega$ *for all limited* x *and* $y \simeq 0$.

Proof Analogous to the proof of Lemma 5.3. □

Lemma 5.5 *Let* $\lim_{X\to\infty} X(|F_2'(X,\hat{Y}(X))|-1) = r$ *for some* $r \in \mathbb{R}$. *Then under the macroscope* $M_{\omega,\tilde{Y}(\omega)}$ *one has* ${}^\circ\hat{y}_\omega(x) = {}^\circ\tilde{y}_\omega(x) = 0$ *for all positive appreciable* x. *Moreover,* $\hat{y}_\omega$ *is of class* S^1 *for all positive appreciable* x.

Proof Similar to the proof of Lemma 5.2. □

Lemma 5.6 *Let* $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X))-1) = r$ *for some* $r \in \mathbb{R}$. *Then under the macroscope* $M_{\omega,\tilde{Y}(\omega)}$ *one has* $(\partial f_\omega/\partial y)(x,y) = 1 + r/\omega x + \oslash/\omega$ *for all positive appreciable* x *and* $y \simeq 0$.

Proof We have

$$\begin{aligned}&\frac{\partial F}{\partial Y}(\omega x,(1+y)\tilde{Y}(\omega))\\ &\quad = 1 + \frac{\frac{\partial F}{\partial Y}(\omega x,(1+y)\tilde{Y}(\omega))-1}{\frac{\partial F}{\partial Y}(\omega x,\hat{Y}(\omega x))-1}\cdot\left(\frac{\partial F}{\partial Y}(\omega x,\hat{Y}(\omega x))-1\right).\end{aligned}$$

By Lemma 5.5 and formula (1) we obtain for all positive appreciable x and $y \simeq 0$ that

$$\frac{\partial f_\omega}{\partial y}(x,y) = 1+(1+\oslash)\left(\frac{r}{\omega x}+\frac{\oslash}{\omega}\right) = 1+\frac{r}{\omega x}+\frac{\oslash}{\omega}.$$

□

Lemma 5.7 *Let* $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X))+1) = -r$ *for some* $r \in \mathbb{R}$. *Then under the macroscope* $M_{\omega,\tilde{Y}(\omega)}$ *one has* $(\partial f_\omega/\partial y)(x,y) = -1-r/\omega x+\oslash/\omega$ *for all positive appreciable* x *and* $y \simeq 0$.

Proof Similar to the proof of Lemma 5.6. □

6 Proofs of the characterization theorems

We start with some notation and some conventions used in all of the proofs. We let ω be an arbitrary unlimited integer. We assume always that Φ and Ψ are solutions such that $\Phi(\omega) = (1+\oslash)\Psi(\omega) = (1+\oslash)\tilde{Y}(\omega)$ and $\Phi(\omega) \neq \Psi(\omega)$, $\Phi(\omega) \neq \tilde{Y}(\omega)$, with the exception of the sufficient part of the proof of the strongly attractive case, where we assume that $\Phi(\omega+x) = (1+\oslash)\Psi(\omega+x) = (1+\oslash)\tilde{Y}(\omega+x)$ and $\Phi(\omega+x) \neq \Psi(\omega+x)$ for some limited $x \geq 0$. Let $X \geq \omega$ such that still $\Phi(X) =$

$(1+\varnothing)\Psi(X) = (1+\varnothing)\tilde{Y}(X)$ and $\Phi(X) \neq \Psi(X)$. In the case of telescopes $T_{\omega,\tilde{Y},L}$ we define d_ω and δ_ω by

$$d_\omega(x) = \frac{\varphi_\omega(x) - \tilde{y}_\omega(x)}{\varphi_\omega(0) - \tilde{y}_\omega(0)}, \quad \delta_\omega(x) = \frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(0) - \psi_\omega(0)}.$$

With an abuse of language, we use the same notation for differences of solutions, when rescaled by macroscopes $M_{\omega,\tilde{Y}(\omega)}$; by definition

$$d_\omega(x) = \frac{\varphi_\omega(x) - \tilde{y}_\omega(x)}{\varphi_\omega(1) - \tilde{y}_\omega(1)}, \quad \delta_\omega(x) = \frac{\varphi_\omega(x) - \psi_\omega(x)}{\varphi_\omega(1) - \psi_\omega(1)}.$$

Note that for some $\eta \simeq 0$

$$\frac{\Phi(X+1) - \Psi(X+1)}{\Phi(X) - \Psi(X)} = F_2'(X, (1+\eta)\hat{Y}(X)). \tag{19}$$

This implies that under the telescopes $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ one has

$$\frac{\delta_\omega(x+\gamma_\omega)}{\delta_\omega(x)} = \frac{\partial f_\omega(x, (1+\eta)(1+\hat{y}_\omega(x)) - 1)}{\partial y} \tag{20}$$

and similarly under the macroscopes $M_{\omega,\tilde{Y}(\omega)}$ one has

$$\frac{\delta_\omega(x+1/\omega)}{\delta_\omega(x)} = \frac{\partial f_\omega(x, (1+\eta)(1+\hat{y}_\omega(x)) - 1)}{\partial y}. \tag{21}$$

If we take $x = 1$ in (21), we find

$$\delta_\omega(1+1/\omega) = \frac{\partial f_\omega(1, \theta)}{\partial y}, \tag{22}$$

where $\theta \simeq 0$.

Proof of Theorem 3.2.1. Let $\lim_{X\to\infty} \hat{Y}(X+1)/\hat{Y}(X) = 1$ and $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 0$. By Lemma 5.1 under the telescope $T_{\omega,\tilde{Y}(\omega),1}$ one has ${}^\circ\tilde{y}_\omega = 0$. Let $x \in \mathbb{N}$ be limited. By (20) and (1)

$$\frac{\varphi_\omega(x+1) - \psi_\omega(x+1)}{\varphi_\omega(x) - \psi_\omega(x)} \simeq 0.$$

Hence $\tilde{Y}$ is a strongly attractive river.

Conversely, if $\tilde{Y}$ is a strongly attractive river, we have ${}^{\circ}\tilde{y}_\omega(1) = 0$. So $\tilde{Y}(\omega + 1)/\tilde{Y}(\omega) \simeq 1$, hence $\hat{Y}(\omega+1)/\hat{Y}(\omega) \simeq 1$ and $\lim_{X\to\infty} \hat{Y}(X+1)/\hat{Y}(X) = 1$. Also $(\varphi_\omega(1) - \psi_\omega(1))\,/(\varphi_\omega(0) - \psi_\omega(0)) \simeq 0$. By formula (19) one has $F_2'(\omega, (1+\eta)\hat{Y}(\omega)) \simeq 0$ for some $\eta \simeq 0$. Then

$$F_2'(\omega, \hat{Y}(\omega)) = F_2'(\omega, \hat{Y}(\omega)) - 1 + 1 = (1+\varnothing)(F_2'(\omega, (1+\eta)\hat{Y}(\omega)) - 1) + 1 = \varnothing.$$

Hence $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 0$. □

Proof of Theorem 3.2.2. Similar to the proof of Theorem 3.2.1, noting that for limited $x \geq 0$

$$\delta_\omega(x) = \prod_{0\leq\xi<x} \frac{\varphi_\omega(\xi+1) - \psi_\omega(\xi+1)}{\varphi_\omega(\xi) - \psi_\omega(\xi)} \simeq a^x.$$

□

Proof of Theorem 3.2.3. Assume $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 1$, $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) - 1) = -\infty$ and that $F_2'(X, \hat{Y}(X)) - 1$ is slowly varying at scale $1/||F_2'(X, \hat{Y}(X))| - 1|$. Firstly, by Lemma 5.2 under the telescope $T_{\omega, \tilde{Y}(\omega), 1/\gamma_\omega}$ one has ${}^{\circ}\tilde{y}_\omega = 0$.

Secondly, we prove that the difference of two solutions which at 0 under the telescope are infinitely close to 0 is of class S^1. Let $x \geq 0$ be limited. Since

$$\frac{d_\omega(x+\gamma_\omega) - d_\omega(x)}{\gamma_\omega} = \frac{d_\omega(x)}{\gamma_\omega}\left(\frac{\varphi_\omega(x+\gamma_\omega) - \tilde{y}_\omega(x+\gamma_\omega)}{\varphi_\omega(x) - \tilde{y}_\omega(x)} - 1\right), \tag{23}$$

we derive from (20) and Lemma 5.3 that as long as $d_\omega(x)$ is limited,

$$\frac{d_\omega(x+\gamma_\omega) - d_\omega(x)}{\gamma_\omega} = \frac{d_\omega(x)}{\gamma_\omega}(1 + (1+\varnothing)g_\omega - 1) = -d_\omega(x) + \varnothing. \tag{24}$$

By the Stroboscopy Lemma $d_\omega(x) \simeq e^{-x}$ for such x, which also implies that in fact $d_\omega(x)$ is limited for all limited x. This means that every solution which under the telescope is infinitely close to zero at $x = 0$ is infinitely close to zero for all limited $x \geq 0$. Hence $\delta_\omega(x)$ satisfies (24) for all limited $x \geq 0$, i.e.,

$$\frac{\delta_\omega(x+\gamma_\omega) - \delta_\omega(x)}{\gamma_\omega} \simeq -\delta_\omega(x) \simeq -e^{-x}.$$

This implies that $\delta_\omega(x)$ is of class S^1 for all limited $x \geq 0$.

Thirdly, we prove that $\tilde{y}_\omega$ is also of class S^1 for all limited $x \geq 0$. Note that it satisfies the equation

$$\begin{aligned}\tilde{y}_\omega(x+\gamma_\omega)-\tilde{y}_\omega(x) = &\left(\frac{f_\omega(x,\tilde{y}_\omega(x))-f_\omega(x,\hat{y}_\omega(x))}{\tilde{y}_\omega(x)-\hat{y}_\omega(x)}-1\right)\tilde{y}_\omega(x)+\\ &-\left(\frac{f_\omega(x,\tilde{y}_\omega(x))-f_\omega(x,\hat{y}_\omega(x))}{\tilde{y}_\omega(x)-\hat{y}_\omega(x)}-1\right)\hat{y}_\omega(x)\\ &+\frac{f_\omega(x,\hat{y}_\omega(x))-\hat{y}_\omega(x)}{\hat{y}_\omega(x)(\partial f_\omega(x,\hat{y}_\omega(x))/\partial y-1)}\hat{y}_\omega(x)(\partial f_\omega(x,\hat{y}_\omega(x))/\partial y-1).\end{aligned}$$

Now $\hat{y}_\omega(x) \simeq 0$ by Lemma 5.2. Then for some $\beta \simeq 0$

$$\frac{f_\omega(x,\tilde{y}_\omega(x))-f_\omega(x,\hat{y}_\omega(x))}{\tilde{y}_\omega(x)-\hat{y}_\omega(x)}-1=\frac{\partial f_\omega(x,\beta)}{\partial y}-1,$$

which by Lemma 5.3 is of the form $(1+\varnothing)g_\omega$. It follows also from Lemma 5.3 that $\partial f_\omega(x,\hat{y}_\omega(x))/\partial y - 1 = (1+\varnothing)g_\omega$. Definition 2.2.2 implies that

$$\frac{f_\omega(x,\hat{y}_\omega(x))-\hat{y}_\omega(x)}{\hat{y}_\omega(x)(\partial f_\omega(x,\hat{y}_\omega(x))/\partial y-1)} \simeq 0.$$

By these estimations the difference $\tilde{y}_\omega(x+\gamma_\omega)-\tilde{y}_\omega(x)$ takes the form

$$\tilde{y}_\omega(x+\gamma_\omega)-\tilde{y}_\omega(x)=(1+\varnothing)g_\omega\tilde{y}_\omega(x)-(1+\varnothing)g_\omega\cdot\varnothing+\varnothing\cdot\varnothing(1+\varnothing)g_\omega.$$

So

$$\frac{\tilde{y}_\omega(x+\gamma_\omega)-\tilde{y}_\omega(x)}{\gamma_\omega}\simeq-\tilde{y}_\omega(x)\simeq 0, \tag{25}$$

hence $\tilde{y}_\omega$ is of class S^1. Combining, we conclude that $\tilde{Y}$ is a weakly exponentially attractive river of class S^1.

Conversely, we remark first that because $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ is a telescope, one has $1/\gamma_\omega = \varnothing\omega$, and because ω is arbitrary

$$\lim_{X\to\infty} X\cdot||F_2'(X,\hat{Y}(X))|-1|=+\infty. \tag{26}$$

Next, because $\delta_\omega(x)$ is of class S^1, in particular ${}^{o}\delta_\omega(x) = e^{-x}$ for $x \geq 0$, which implies that the steps γ_ω of the discrete function δ_ω must be infinitesimal. Hence $\lim_{X\to\infty}|F_2'(X,\hat{Y}(X))|-1=0$. Let $x \geq 0$ be limited. Then

$$\frac{\delta_\omega(x+\gamma_\omega)-\delta_\omega(x)}{\gamma_\omega}\simeq\frac{de^{-x}}{dx}=-e^{-x}\simeq-\delta_\omega(x).$$

Hence

$$\delta_\omega(x+\gamma_\omega)/\delta_\omega(x)=1-(1+\varnothing)\gamma_\omega. \tag{27}$$

Notice that $\hat{y}_\omega(x) \simeq 0$, because

$$\hat{y}_\omega(x) = \frac{\hat{Y}(\omega + x/\gamma_\omega)}{\tilde{Y}(\omega + x/\gamma_\omega)}(1 + \tilde{y}_\omega(x)) - 1 = (1+\oslash)(1+\oslash) - 1 = \oslash. \tag{28}$$

Then by (20) there exists $\bar{\eta} \simeq 0$ such that $\partial f_\omega(x, \bar{\eta})/\partial y = \delta_\omega(x+\gamma_\omega)/\delta_\omega(x)$. Hence it follows from formula (1) and formula (27), applied for $x = 0$, that $\partial f_\omega(0,0)/\partial y = 1 - (1+\oslash)\gamma_\omega$. Because $\gamma_\omega \simeq 0$ and $\gamma_\omega > 0$ we have $F_2'(\omega, \hat{Y}(\omega)) \simeq 1$, $F_2'(\omega, \hat{Y}(\omega)) < 1$. From this and (26) we derive that $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = 1$ and $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) - 1) = -\infty$. Finally we prove that $F_2'(X, \hat{Y}(X)) - 1$ is slowly varying at scale $1/||F_2'(X, \hat{Y}(X))| - 1|$. Noting that $\hat{y}_\omega(x) \simeq 0$ we derive as above from (20), (1) and (27) that $\partial f(x, \hat{y}_\omega(x))/\partial y = 1 - (1+\oslash)\gamma_\omega$ for all limited $x \geq 0$. Hence $\partial f(x, \hat{y}_\omega(x))/\partial y - 1 = (1+\oslash)(\partial f_\omega(0,0)/\partial y - 1)$ and

$$\begin{aligned} &F_2'\left(\omega + \frac{x}{||F_2'(\omega, \hat{Y}(\omega))| - 1|}, \hat{Y}\left(\omega + \frac{x}{||F_2'(\omega, \hat{Y}(\omega))| - 1|}\right)\right) - 1 \\ &= (1+\oslash)(F_2'(\omega, \hat{Y}(\omega)) - 1), \end{aligned}$$

so $F_2'(X, \hat{Y}(X)) - 1$ is slowly varying at scale $1/||F_2'(X, \hat{Y}(X))| - 1|$ by formula (2).

Combining, we see that the three conditions of Theorem 3.2.3 are verified. □

Proof of Theorem 3.2.4. Assume $\lim_{X\to\infty} F_2'(X, \hat{Y}(X)) = -1$, $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) + 1) = +\infty$ and that $F_2'(X, \hat{Y}(X)) + 1$ is slowly varying at scale $1/||F_2'(X, \hat{Y}(X))| - 1|$. Firstly, by Lemma 5.2 under the telescope $T_{\omega, \tilde{Y}(\omega), 1/\gamma_\omega}$ one has ${}^\circ\tilde{y}_\omega = 0$.

Secondly, we prove that the difference of two solutions which at 0 under the telescope are infinitely close to 0 is of class $|S|^1$, but not of class S^1. Let $x \geq 0$ be limited. We assume first that x/γ_ω is even. Since

$$\frac{d_\omega(x+\gamma_\omega) + d_\omega(x)}{\gamma_\omega} = \frac{d_\omega(x)}{\gamma_\omega}\left(\frac{\varphi_\omega(x+\gamma_\omega) - \tilde{y}_\omega(x+\gamma_\omega)}{\varphi_\omega(x) - \tilde{y}_\omega(x)} + 1\right), \tag{29}$$

we derive from (20) and Lemma 5.4 that as long as $d_\omega(x)$ is limited,

$$\frac{d_\omega(x+\gamma_\omega) + d_\omega(x)}{\gamma_\omega} = \frac{d_\omega(x)}{\gamma_\omega}(-1 - (1+\oslash)g_\omega + 1) = d_\omega(x)(1+\oslash). \tag{30}$$

Hence $d_\omega(x+\gamma_\omega) = -d_\omega(x)(1 - (1+\oslash)\gamma_\omega)$. If x/γ_ω is even, formula (30) implies that

$$\frac{A_{d_\omega}(x+\gamma_\omega) - A_{d_\omega}(x)}{\gamma_\omega} = \frac{-d_\omega(x+\gamma_\omega) - d_\omega(x)}{\gamma_\omega} \simeq -d_\omega(x) = -A_{d_\omega}(x),$$

and if x/γ_ω is odd, also

$$\frac{A_{d_\omega}(x+\gamma_\omega)-A_{d_\omega}(x)}{\gamma_\omega}=\frac{d_\omega(x+\gamma_\omega)+d_\omega(x)}{\gamma_\omega}\simeq d_\omega(x)=-A_{d_\omega}(x).$$

Hence by the Stroboscopy Lemma $A_{d_\omega}(x)\simeq e^{-x}$ for such x, which also implies that $|d_\omega(x)|$ is limited for all limited x. This means that every solution which under the telescope is infinitely close to zero at $x=0$ is infinitely close to zero for all limited $x\geq 0$. Then we may apply the formulae above also to A_{δ_ω}, and find that for all limited $x\geq 0$

$$\frac{A_{\delta_\omega}(x+\gamma_\omega)-A_{\delta_\omega}(x)}{\gamma_\omega}\simeq -A_{\delta_\omega}(x)\simeq -e^{-x}. \tag{31}$$

Hence δ_ω is of class $|S|^1$ for all limited $x\geq 0$. Notice that it follows from (31) that

$$\frac{\delta_\omega(\gamma_\omega)-\delta_\omega(0)}{\gamma_\omega}\simeq -\frac{2}{\gamma_\omega}+1,$$

which is unlimited. Hence δ_ω is of not of class S^1.

Thirdly, we prove that $\tilde{y}_\omega-\hat{y}_\omega$ is of class $|S|^1$ for all limited $x\geq 0$. The discrete function $\tilde{y}_\omega-\hat{y}_\omega$ satisfies the equation

$$\begin{aligned}&\tilde{y}_\omega(x+\gamma_\omega)-\hat{y}_\omega(x+\gamma_\omega)+\tilde{y}_\omega(x)-\hat{y}_\omega(x)\\&=\left(\frac{f_\omega(x,\tilde{y}_\omega(x))-f_\omega(x,\hat{y}_\omega(x))}{\tilde{y}_\omega(x)-\hat{y}_\omega(x)}+1\right)(\tilde{y}_\omega(x)-\hat{y}_\omega(x))\\&\quad+\frac{f_\omega(x,\hat{y}_\omega(x))-\hat{y}_\omega(x)}{\hat{y}_\omega(x)(\partial f_\omega(x,\hat{y}_\omega(x))/\partial y+1)}\hat{y}_\omega(x)(\partial f_\omega(x,\hat{y}_\omega(x))/\partial y+1)+\\&\quad-(\hat{y}_\omega(x+\gamma_\omega)-\hat{y}_\omega(x)).\end{aligned} \tag{32}$$

By Lemma 5.2 we have $\hat{y}_\omega(x+\gamma_\omega)-\hat{y}_\omega(x)=\varnothing\gamma_\omega$ and $\hat{y}_\omega(x)\simeq 0$. The latter implies that for some $\beta\simeq 0$

$$\frac{f_\omega(x,\tilde{y}_\omega(x))-f_\omega(x,\hat{y}_\omega(x))}{\tilde{y}_\omega(x)-\hat{y}_\omega(x)}+1=\frac{\partial f_\omega(x,\beta)}{\partial y}+1,$$

which by Lemma 5.4 is of the form $-(1+\varnothing)g_\omega$. It follows also from Lemma 5.4 that $\partial f_\omega(x,\hat{y}_\omega(x)/\partial y+1=-(1+\varnothing)g_\omega$. Definition 2.2.2 implies that

$$\frac{f_\omega(x,\hat{y}_\omega(x))-\hat{y}_\omega(x)}{\hat{y}_\omega(x)(\partial f_\omega(x,\hat{y}_\omega(x))/\partial y+1)}\simeq 0.$$

By these estimations equation (32) takes the form

$$\begin{aligned}&\tilde{y}_\omega(x+\gamma_\omega)-\hat{y}_\omega(x+\gamma_\omega)+\tilde{y}_\omega(x)-\hat{y}_\omega(x)=\\&\quad-(1+\varnothing)g_\omega(\tilde{y}_\omega(x)-\hat{y}_\omega(x))-(1+\varnothing)g_\omega\cdot\varnothing+\varnothing\cdot\varnothing(1+\varnothing)g_\omega.\end{aligned}$$

So

$$\frac{\tilde{y}_\omega(x+\gamma_\omega)-\hat{y}_\omega(x+\gamma_\omega)+\tilde{y}_\omega(x)-\hat{y}_\omega(x)}{\gamma_\omega}\simeq \tilde{y}_\omega(x)-\hat{y}_\omega(x)\simeq 0, \tag{33}$$

hence $\tilde{y}_\omega-\hat{y}_\omega$ is of class $|S|^1$. Combining, we conclude that $\tilde{Y}$ is a weakly exponentially attractive river of class $|S|^1$, but not of class S^1.

The proof of the converse follows the lines of the converse part of the proof of Theorem 3.2.3. We remark first that because $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$ is a telescope, one has $1/\gamma_\omega \simeq \varnothing\omega$, so (26) holds. Because $\delta_\omega(x)$ is of class $|S|^1$ in particular ${}^{\circ}|\delta_\omega(x)|=e^{-x}$ for $x\geq 0$, which implies that the steps γ_ω of the discrete function δ_ω must be infinitesimal. Hence $\lim_{X\to\infty}|F_2'(X,\hat{Y}(X))|-1=0$. Let $x\geq 0$ be limited. Then

$$\frac{A_{\delta_\omega}(x+\gamma_\omega)-A_{\delta_\omega}(x)}{\gamma_\omega}\simeq\frac{de^{-x}}{dx}=-e^{-x}\simeq -A_{\delta_\omega}(x).$$

Hence

$$\delta_\omega(x+\gamma_\omega)/\delta_\omega(x)=-1+(1+\varnothing)\gamma_\omega. \tag{34}$$

One shows as in (28) that $\hat{y}_\omega(x)\simeq 0$. Then by (20) there exists $\bar{\eta}\simeq 0$ such that $\partial f_\omega(x,\bar{\eta})/\partial y=\delta_\omega(x+\gamma_\omega)/\delta_\omega(x)$. Hence it follows from formula (1) and formula (34), applied for $x=0$, that $\partial f_\omega(0,0)/\partial y=-1+(1+\varnothing)\gamma_\omega$. Because $\gamma_\omega\simeq 0$ and $\gamma_\omega>0$ we have $F_2'(\omega,\hat{Y}(\omega))\simeq -1$, $F_2'(\omega,\hat{Y}(\omega))>-1$. From this and (26) we derive that $\lim_{X\to\infty}F_2'(X,\hat{Y}(X))=-1$ and $\lim_{X\to\infty}X(F_2'(X,\hat{Y}(X))+1)=+\infty$. Finally we prove that $F_2'(X,\hat{Y}(X))+1$ is slowly varying at scale $1/||F_2'(X,\hat{Y}(X))|-1|$. Noting that $\hat{y}_\omega(x)\simeq 0$ for all limited $x\geq 0$ we derive as above from formula (20), formula (1) and formula (34) that $\partial f(x,\hat{y}_\omega(x))/\partial y=-1+(1+\varnothing)\gamma_\omega$ for all limited $x\geq 0$. Hence

$$F_2'\left(\omega+\frac{x}{||F_2'(\omega,\hat{Y}(\omega))|-1|},\hat{Y}\left(\omega+\frac{x}{||F_2'(\omega,\hat{Y}(\omega))|-1|}\right)\right)+1$$
$$=(1+\varnothing)(F_2'(\omega,\hat{Y}(\omega))+1),$$

so $F_2'(X,\hat{Y}(X))+1$ is slowly varying at scale $1/||F_2'(X,\hat{Y}(X))|-1|$ by formula (2).

Combining, we see that the three conditions of Theorem 3.2.4 are verified. □

Proof of Theorem 3.2.5. The proof is similar to the proof of Theorem 3.2.3. Let $r<0$ be such that $\lim_{X\to\infty}X(F_2'(X,\hat{Y}(X))-1)=r$. Firstly, by Lemma 5.5 under the macroscope $M_{\omega,\tilde{Y}(\omega)}$ one has ${}^{\circ}\tilde{y}_\omega=0$.

Secondly, we prove that δ_ω is of class S^1 for limited $x \geq 1$. We may adapt slightly the argument of the proof of Theorem 3.2.3 leading to formula (24), now using (21) and Lemma 5.6, to obtain that as long as $d_\omega(x)$ is limited,

$$\frac{d_\omega(x+1/\omega)-d_\omega(x)}{1/\omega} = \frac{d_\omega(x)}{1/\omega}\left(1+\frac{r}{\omega x}+\frac{\varnothing}{\omega}-1\right) = \frac{r}{x}d_\omega(x)+\varnothing. \tag{35}$$

By the Stroboscopy Lemma $d_\omega(x) \simeq x^r$ for such x, which implies that in fact $d_\omega(x)$ is limited for all limited $x \geq 1$. This means that every solution which under the macroscope is infinitely close to zero at $x = 1$ is infinitely close to zero for all limited $x \geq 1$. Hence $\delta_\omega(x)$ satisfies (35) for all limited $x \geq 1$, i.e.,

$$\frac{\delta_\omega(x+1/\omega)-\delta_\omega(x)}{1/\omega} \simeq \frac{r}{x}\delta_\omega(x) \simeq rx^{r-1}.$$

This implies that $\delta_\omega(x)$ is of class S^1 for all limited $x \geq 1$.

Thirdly, we prove that $\tilde{y}_\omega$ is of class S^1 for limited $x \geq 1$. A slight adaptation of the argument used in the proof of Theorem 3.2.3 leading to formula (25) yields the estimation

$$\frac{\tilde{y}_\omega(x+1/\omega)-\tilde{y}_\omega(x)}{1/\omega} \simeq \frac{r}{x}\tilde{y}_\omega(x) \simeq 0.$$

Hence $\tilde{y}_\omega$ is of class S^1 for $x \geq 1$. Combining, we conclude that $\tilde{Y}$ is a polynomially attractive river of class S^1.

Conversely, assume under the macroscope $M_{\omega,\tilde{Y}(\omega)}$ one has ${}^{o}\delta_\omega(x) = x^r$ for $x \geq 1$, with $r < 0$. Because δ_ω is of class S^1

$$\frac{\delta_\omega(x+1/\omega)-\delta_\omega(x)}{1/\omega} \simeq \frac{dx^r}{dx} \simeq \frac{r}{x}\delta_\omega(x).$$

Hence

$$\delta_\omega(1+1/\omega) = 1+\frac{r}{\omega}+\frac{\varnothing}{\omega}. \tag{36}$$

By (22) there exists $\theta \simeq 0$ such that $\partial f_\omega(1,\theta)/\partial y = \delta_\omega(1+1/\omega)$. Then it follows from formula (1) and formula (34) that $\partial f_\omega(1,0)/\partial y = 1+\frac{r}{\omega}+\frac{\varnothing}{\omega}$. Hence $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X))-1) = r$. □

Proof of Theorem 3.2.6. We follow the lines of the proof of Theorem 3.2.4, adapting it as has been done in the proof of Theorem 3.2.5. Assume $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X))+1) = -r$ with $r > 0$.

Firstly, by Lemma 5.5 under the macroscope $M_{\omega,\tilde{Y}(\omega)}$ one has ${}^{o}\tilde{y}_\omega = 0$.

Secondly, in order to prove that the difference of two solutions which at 1 under the macroscope are infinitely close to 0 is of class $|S|^1$, but not of class S^1, by an appropriate adaptation of the argument in the proof of Theorem 3.2.4 leading to (31) one obtains for limited $x \geq 1$ that $A_{\delta_\omega}(x) \simeq x^{-r}$ and

$$\frac{A_{\delta_\omega}(x+1/\omega)-A_{\delta_\omega}(x)}{1/\omega} \simeq \frac{r}{x}A_{\delta_\omega}(x) \simeq rx^{-r-1}. \tag{37}$$

This implies that $\delta_\omega(x)$ is of class $|S|^1$ for all limited $x \geq 1$. Notice that it follows from (37) that

$$\frac{\delta_\omega(1+1/\omega)-\delta_\omega(1)}{1/\omega} \simeq -2\omega + r,$$

which is unlimited. Hence δ_ω is of not of class S^1.

Thirdly, we prove that $\tilde{y}_\omega - \hat{y}_\omega$ is of class $|S|^1$ for limited $x \geq 1$. A slight adaptation of the argument in the proof of Theorem 3.2.4 leading to (33) shows that

$$\frac{\tilde{y}_\omega(x+1/\omega)-\hat{y}_\omega(x+1/\omega)+\tilde{y}_\omega(x)-\hat{y}_\omega(x)}{1/\omega} \simeq \frac{r}{x}(\tilde{y}_\omega(x)-\hat{y}_\omega(x)) \simeq 0,$$

hence $\tilde{y}_\omega(x) - \hat{y}_\omega(x)$ is of class $|S|^1$ for limited $x \geq 1$. Combining, we conclude that $\tilde{Y}$ is a polynomially attractive river of class $|S|^1$.

Conversely, assume under the macroscope $M_{\omega,\tilde{Y}(\omega)}$ one has $^\circ|\delta_\omega(x)| = x^{-r}$ for $x \geq 1$, with $r > 0$. Because δ_ω is of class $|S|^1$,

$$\frac{A_{\delta_\omega}(x+1/\omega)-A_{\delta_\omega}(x)}{1/\omega} \simeq \frac{dx^{-r}}{dx} \simeq -\frac{r}{x}A_{\delta_\omega}(x).$$

Hence

$$\frac{\delta_\omega(1+1/\omega)}{1/\omega} = -1 + \frac{r}{\omega} + \frac{\varnothing}{\omega}. \tag{38}$$

By (22) there exists $\eta \simeq 0$ such that $\partial f_\omega(1,\eta)/\partial y = \delta_\omega(1+1/\omega)$. Then it follows from formula (1) and formula (38) that $\partial f_\omega(1,0)/\partial y = (1+\varnothing)\,(\delta_\omega(1+1/\omega)) = -1 + \frac{r}{\omega} + \frac{\varnothing}{\omega}$. Hence $\lim_{X\to\infty} X(F_2'(X,\hat{Y}(X))+1) = r$. □

Finally we consider the proof of Theorem 3.4. We prove first a lemma.

Lemma 6.1 *Let (D) be a standard difference equation, $\hat{Y}$ be a standard approximate solution and $\tilde{Y}$ be a standard solution asymptotic to $\hat{Y}$. Then there exist standard A_0, B_0 with $B_0 > 0$ such that $(|F_2'(X,Y)|-1)/(|F_2'(X,\tilde{Y}(X))|-1)$ is appreciable on $\{V \equiv (X,Y) \mid X \geq A_0, (1-B_0)\tilde{Y}(X) \leq Y \leq (1+B_0)\tilde{Y}(X)\}$.*

Proof We may apply (1) to $\tilde{Y}$ as well as to $\hat{Y}$ and weaken it to obtain that $(|F_2'(X,Y)| - 1)/(|F_2'(X,\tilde{Y}(X))|-1) = @$ for all $X \simeq \infty$ and Y such that $Y/\tilde{Y}(X) \simeq 1$. Applying permanence in the same manner as in the proof of Theorem 4.3 we derive that $(|F_2'(X,Y)|-1)/(|F_2'(X,\tilde{Y}(X))|-1) = @$ on some set standard set V of the above form. □

Proof of Theorem 3.4. Firstly, we prove that $\tilde{Y}$ is a drain if and only if there exists some standard C such that $\sum_{X\geq C}|F_2'(X,\hat{Y}(X))|-1$ converges.

Assume the latter property holds. Now $|F_2'(X,\hat{Y}(X))| > 1$ for all $X \simeq \infty$ or $|F_2'(X,\hat{Y}(X))| < 1$ for all $X \simeq \infty$ by Definition 2.2.1. So $\sum_{X\geq C}\gamma_X$ converges.

Then $\gamma_X \simeq 0$ for all $X \simeq \infty$. Let $\omega, \xi \simeq \infty$ with $\xi > \omega$. Consider the product $\prod_{\omega \le X < \xi} 1 + £\gamma_X$. We have

$$\prod_{\omega \le X < \xi} 1 + £\gamma_X = \exp \sum_{\omega \le X < \xi} \log(1 + £\gamma_X) = \exp £ \sum_{\omega \le X < \xi} \gamma_X = \exp \varnothing = 1 + \varnothing. \tag{39}$$

By Definition 2.2.3 we have $\hat{Y}(X+1)/\hat{Y}(X) = 1 + \varnothing\gamma_X$, so by (39)

$$\frac{\hat{Y}(\xi)}{\hat{Y}(\omega)} = \prod_{\omega \le X < \xi} \frac{\hat{Y}(X+1)}{\hat{Y}(X)} = \prod_{\omega \le X < \xi} 1 + \varnothing\gamma_X = 1 + \varnothing.$$

Because $\hat{Y}$ is standard we derive from the nonstandard characterization of the convergence of Cauchy sequences that $\lim_{X\to\infty} \hat{Y}(X) = D$ for some (standard) $D \neq 0$. Then also $\lim_{X\to\infty} \tilde{Y}(X) = D$.

Let B_0 be as in the above lemma. Let $\bar{Y}$ be another solution such that $|\bar{Y}(\omega) - \tilde{Y}(\omega)| \le B_0/2$. We show that $|\bar{Y}(X) - \tilde{Y}(X)| \simeq |\bar{Y}(\omega) - \tilde{Y}(\omega)|$ for all $X \simeq \infty$. Assume $X \ge \omega$ is such that still $|\bar{Y}(X) - \tilde{Y}(X)| \le B_0$. By (19) and the lemma, for all Z with $\omega \le Z \le X$

$$\left|\frac{\bar{Y}(Z+1) - \tilde{Y}(Z+1)}{\bar{Y}(Z) - \tilde{Y}(Z)}\right| = 1 + @\gamma_Z.$$

By (39) in fact $|\bar{Y}(X+1) - \tilde{Y}(X+1)| = (1+\varnothing)|\bar{Y}(\omega) - \tilde{Y}(\omega)|$. Then $|\bar{Y}(X) - \tilde{Y}(X)| \simeq |\bar{Y}(\omega) - \tilde{Y}(\omega)|$ for all $X \ge \omega$ by nested induction [6, Lemma 3.1]. This implies that $|\bar{Y}(X) - \tilde{Y}(X)| \simeq |\bar{Y}(\omega) - \tilde{Y}(\omega)|$ also for all $X \simeq \infty$, $X < \omega$ as long as $|\bar{Y}(X) - \tilde{Y}(X)| \le B_0/2$, if not the above argument applied to X and $\omega > X$ yields $|\bar{Y}(\omega) - \tilde{Y}(\omega)| \simeq |\bar{Y}(X) - \tilde{Y}(X)| \not\simeq |\bar{Y}(\omega) - \tilde{Y}(\omega)|$. If we take $\bar{Y}$ such that $\bar{Y}(\omega) \simeq \tilde{Y}(\omega)$, we see that $\tilde{Y}$ is a drain; indeed for all $X \simeq \infty$ the near-equality $\bar{Y}(X) \simeq \tilde{Y}(X)$ implies $\bar{Y}(X)/\tilde{Y}(X) \simeq 1$, since $\tilde{Y}(X) \simeq D \not\simeq 0$.

Conversely, let $\omega \simeq \infty$ and $\bar{Y}$ be a solution such that $\bar{Y}(\omega) \neq \tilde{Y}(\omega)$, with $\bar{Y}(\omega)/\tilde{Y}(\omega) \simeq 1$. Let $\xi > \omega$ be arbitrary. By (19) and (1), for all X with $\xi \le X < \omega$

$$1 \simeq \left|\frac{\bar{Y}(X+1) - \tilde{Y}(X+1)}{\bar{Y}(X) - \tilde{Y}(X)}\right| = 1 + (1+\varnothing)g_X.$$

So $g_X \simeq 0$ for all X with $\xi \le X < \omega$. Also

$$1 \simeq \left|\frac{\bar{Y}(\xi) - \tilde{Y}(\xi)}{\bar{Y}(\omega) - \tilde{Y}(\omega)}\right| = \prod_{\omega \le X < \xi} 1 + (1+\varnothing)g_X = \exp(1+\varnothing) \sum_{\omega \le X < \xi} g_X.$$

Then $\sum_{\omega \le X < \xi} g_X \simeq 0$ and by the nonstandard version of the Cauchy characterization of convergence $\sum_{X \ge C} g_X$ converges for some standard C.

In order to prove that the drain $\tilde{Y}$ is of class S^1 if and only if $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) - 1) = 0$ one may adapt in an obvious way the corresponding part of the proof of Theorem 3.2.5 to the case $r = 0$; note that if $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) - 1) = 0$, the sum $\sum_{X\geq C} g_X$ is transformed into $\sum_{X\geq C}(F_2'(X, \hat{Y}(X)) - 1)$, which thus converges. Similarly, in order to prove that the drain $\tilde{Y}$ is of class $|S|^1$, but not of class S^1, if and only if $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) + 1) = 0$, one may adapt in an obvious way the corresponding part of the proof of Theorem 3.2.6 to the case $r = 0$; finally, if $\lim_{X\to\infty} X(F_2'(X, \hat{Y}(X)) + 1) = 0$, the convergence of $\sum_{X\geq C} g_X$ implies the convergence of $\sum_{X\geq C}(F_2'(X, \hat{Y}(X)) + 1)$. □

7 Further remarks

7.1 Examples of rivers of quadratic equations

In [7] some quadratic equations were solved for approximate solutions and solutions asymptotic to them. We investigate here the nature of these solutions.

Example 7.1 Consider the equation

$$Y(X+1) = Y(X)^2 - X^a. \tag{40}$$

1. $a > 0$. The sequences $\hat{Y}_1(X) = X^{a/2}$ and $\hat{Y}_2(X) = -X^{a/2}$ are solutions of the associated asymptotic functional equation (A). By the Existence Theorem the equation (40) has a solution $\tilde{Y}_1(X) \sim X^{a/2}$ for $X \to \infty$ and a solution $\tilde{Y}_2(X) \sim -X^{a/2}$ for $X \to \infty$. One has $F_2'(X, \hat{Y}_1(X)) = 2X^{a/2}$ and $F_2'(X, \hat{Y}_2(X)) = -2X^{a/2}$. Hence both solutions are strongly repulsive rivers.

2. $a < 0$. The sequences $\hat{Y}_1(X) = 1$ and $\hat{Y}_2(X) = -X^a$ are solutions of the associated asymptotic functional equation (A). By the Existence Theorem the equation (40) has a solution $\tilde{Y}_1(X)$ with $\lim_{X\to\infty} \tilde{Y}_1(X) = 1$ and a solution $\tilde{Y}_2(X) \sim -X^a$ for $X \to \infty$. One has $F_2'(X, \hat{Y}_1(X)) = 2$, so $\tilde{Y}_1$ is a moderately repulsive river. Further $F_2'(X, \hat{Y}_2(X)) = -2X^a$. Hence $\tilde{Y}_2$ is a strongly attractive river.

Example 7.2 Consider the equation

$$Y(X+1) = Y(X)^2 + Y(X) - X^a. \tag{41}$$

The sequences $\hat{Y}_1(X) = X^{a/2}$ and $\hat{Y}_2(X) = -X^{a/2}$ are obvious solutions of the associated asymptotic functional equation (A). We have $F_2'(X, \hat{Y}_1(X)) = 2X^{a/2} + 1$ and $F_2'(X, \hat{Y}_2(X)) = -2X^{a/2} + 1$. Also, if $i = 1, 2$,

$$\frac{\hat{Y}_i(X+1) - \hat{Y}_i(X)}{\hat{Y}_i(X)} \sim \frac{a}{2X} \qquad \text{for } X \to \infty.$$

We investigate first the cases where $\hat{Y}_1$ or $\hat{Y}_2$ are not approximate solutions. First,

$$\frac{\hat{Y}_i(X+1)-\hat{Y}_i(X)}{\hat{Y}_i(X)} = o(|F_2'(X,\hat{Y}_i(X))|-1) \qquad \text{for } X \to \infty$$

only for $a > -2$, so $\hat{Y}_1$ or $\hat{Y}_2$ are not approximate solutions for $a \leq -2$. Applying the macroscope $M_{\omega,\omega^{a/2}}$ it has been shown in [7] that indeed there are no solutions asymptotic to $\pm X^{a/2}$ for $X \to \infty$. For $a = 0$, one has $\hat{Y}_2(X) = -1$. Because $F_2'(X,-1) = 1$, the sequence $\hat{Y}_2$ is not an approximate solution, though it is obviously a true solution for $X \geq 1$.

Let $a > -2$. Then we distinguish the following three cases.

1. $-2 < a < 0$. The equation (41) has a weakly exponentially repulsive river $\tilde{Y}_1(X) \sim X^{a/2}$ for $X \to \infty$ of class S^1 and a weakly exponentially attractive river $\tilde{Y}_2(X) \sim -X^{a/2}$ for $X \to \infty$ of class S^1.

2. $a = 0$. The solution $\tilde{Y}_1 = 1$ is a moderate repulsive river. The fact that $F_2'(X,-1) = 1$ suggests that there is little contraction very close to the solution $\tilde{Y}_2 = -1$, still $\tilde{Y}_2 = -1$ is not a drain. Indeed, let Y be a solution and put $D = Y + 1$. Let $X \geq 1$. Then D satisfies $D(X+1) = D(X)(D(X)-1)$ and $D(X+2) = D(X)(1-2D(X)^2(1-D(X)/2))$. Assume $0 < D(X) \leq \frac{1}{2}$. Clearly $0 < D(X+2) < D(X)$, so by induction $0 < D(X+2N) \leq \frac{1}{2}$ for all $N \in \mathbb{N}$. For such N

$$\frac{D(X+2N)}{D(X)} \leq \prod_{0 \leq K < N} \left(1 - \frac{3}{2} D(X+K)^2\right). \tag{42}$$

If $N \simeq \infty$ and $D(X+K) \gtrapprox 0$ for all K with $0 \leq K \leq N$, formula (42) implies that $\frac{D(X+2N)}{D(X)} \simeq 0$, a contradiction. This implies that $D(X+N) \simeq 0$ whenever $N \simeq \infty$. For $X \simeq \infty$, one thus concludes that $\tilde{Y}_2$ is not a drain. Applied to standard X, one concludes in fact that $\tilde{Y}_2$ is asymptotically stable.

3. $a > 0$. There are strongly repulsive rivers $\tilde{Y}_1(X) \sim X^{a/2}$ for $X \to \infty$ and $\tilde{Y}_2(X) \sim -X^{a/2}$ for $X \to \infty$.

7.2 A special class of drains

If a solution $\tilde{Y}$ is a drain, on $H_{\tilde{Y}}$ one observes a sort of almost parallelness, i.e., if $\lim_{X\to\infty} X(|F_2'(X,\hat{Y}(X))|-1) = 0$, every solution $\bar{Y}$ which enters $H_{\tilde{Y}}$ satisfies $\bar{Y}(X) \simeq \tilde{Y}(X)$ for all $X \simeq \infty$. On a microlevel we may still have attraction or repulsion. We investigate here the natural case where $|F_2'(X,\hat{Y}(X))|-1 \sim c/X^s$ for $X \to \infty$, where $c \neq 0$, $s > 1$.

Proposition 7.3 *Let (D) be a standard difference equation and $\hat{Y}$ be a standard approximate solution. Assume for some (standard) $s > 1$ and $c \neq 0$*

$$|F_2'(X,\hat{Y}(X))|-1 \sim c/X^s \quad \textit{for } X \to \infty.$$

Let $\tilde{Y}$ be a standard solution asymptotic to $\hat{Y}$ (i.e., with the same limit as $\hat{Y}$, since $\lim \hat{Y}(X) \neq 0$*). Let $\omega \simeq \infty$ and Φ, Ψ be two solutions such that $\Phi(\omega)/\tilde{Y}(\omega) \simeq \Psi(\omega)/\tilde{Y}(\omega) \simeq 1$ (i.e., such that $\Phi(\omega) \simeq \Psi(\omega) \simeq \tilde{Y}(\omega)$, since $\tilde{Y}(\omega)$ is appreciable). Let $x \geq 1$ be limited. Then*

$$\left|\frac{\Phi(\omega x)-\Psi(\omega x)}{\Phi(\omega)-\Psi(\omega)}\right| = 1+\frac{c+\varnothing}{s-1}\left(1-\frac{1}{x^{s-1}}\right)\cdot\frac{1}{\omega^{s-1}}. \tag{43}$$

Proof Let $1 \leq \xi < x$ be such that $\omega\xi \in \mathbb{N}$. Applying formulae (19) and (1) we find

$$\begin{aligned}\left|\frac{\Phi(\omega\xi+1)-\Psi(\omega\xi+1)}{\Phi(\omega\xi)-\Psi(\omega\xi)}\right| &= (|F_2'(\omega\xi,(1+\varnothing)\hat{Y}(\omega\xi))|-1)+1\\ &= (1+\varnothing)(|F_2'(\omega\xi,\hat{Y}(\omega\xi))|-1)+1\\ &= 1+\frac{c+\varnothing}{\xi^s}\cdot\frac{1}{\omega^s}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{\Phi(\omega x)-\Psi(\omega x)}{\Phi(\omega)-\Psi(\omega)}\right| &= \prod_{1\leq\xi<x}\left|\frac{\Phi(\omega\xi+1)-\Psi(\omega\xi+1)}{\Phi(\omega\xi)-\Psi(\omega\xi)}\right|\\ &= \prod_{1\leq\xi<x} 1+\frac{c+\varnothing}{\xi^s}\cdot\frac{1}{\omega^s}\\ &= \exp\frac{c+\varnothing}{\omega^{s-1}}\sum_{1\leq\xi<x}\frac{1}{\xi^s}\cdot\frac{1}{\omega}\\ &= \exp\frac{c+\varnothing}{\omega^{s-1}}\int_1^x\frac{1}{u^s}\,du\\ &= 1+\frac{c+\varnothing}{s-1}\left(1-\frac{1}{x^{s-1}}\right)\cdot\frac{1}{\omega^{s-1}}.\end{aligned}$$

□

It is not difficult to show that formula (43) holds also for unlimited x, otherwise said, if $\omega, \omega' \simeq \infty$ are such that $\omega'/\omega \simeq \infty$,

$$\left|\frac{\Phi(\omega')-\Psi(\omega')}{\Phi(\omega)-\Psi(\omega)}\right| = 1+\frac{c+\varnothing}{s-1}\cdot\frac{1}{\omega^{s-1}}. \tag{44}$$

7.3 Moving backwards under telescopes and macroscopes

In Definition 2.9.1–3, strong, moderate and exponentially weak rivers $\tilde{Y}$ are defined by specifying the behaviour of $\tilde{Y}$ and neighbouring solutions on the asymptotic halo $H_{\tilde{Y}}$

of $\tilde{Y}$ as viewed by appropriate telescopes $T_{\omega,\tilde{Y},L(\omega)}$, but exclusively for positive limited arguments x. These behaviours are characterized by properties of the partial derivative in the second variable of the function F which defines the difference equation. In fact these behaviours extend to negative limited values of x, as long as we confine ourselves to solutions which for all limited x belong to $H_{\tilde{Y}}$. This follows easily by recentering the telescope from $(\omega, \tilde{Y}(\omega))$ to $(\omega + L(\omega)x, \tilde{Y}(\omega + L(\omega)x))$, and by the Uniqueness Lemma 4.3 (with the exception of strong attraction, where uniqueness is not needed). For instance, under the conditions of Theorem 3.2.2 formula (11) holds for positive and negative limited x, just as formula (12) holds for all limited x under the conditions of Theorem 3.2.3.

A similar remark may be made for polynomial behaviour and drain behaviour under macroscopes for positive appreciable $x < 1$.

7.4 The river and the asymptotic functional equation

Rivers and drains $\tilde{Y}$ may be found in practice by solving the asymptotic functional equation (A), i.e.,

$$\lim_{X\to\infty} \frac{F(X, \hat{Y}(X)) - \hat{Y}(X)}{\hat{Y}(X)(|F_2'(X, \hat{Y}(X))| - 1)} = 0$$

The standard solutions $\tilde{Y}$ may or may not satisfy this equation. Indeed, let $\omega \simeq \infty$. When applied to $\tilde{Y}$ at ω, equation (A) is equivalent to

$$\frac{\tilde{Y}(\omega + 1) - \tilde{Y}(\omega)}{\tilde{Y}(\omega)} = \varnothing \cdot (|F_2'(\omega, \tilde{Y}(\omega))| - 1). \tag{45}$$

If $\tilde{Y}$ is a strong or moderate river, one has $(\tilde{Y}(\omega + 1) - \tilde{Y}(\omega))/\tilde{Y}(\omega) \simeq 0$, while $|F_2'(\omega, \tilde{Y}(\omega))| - 1 \not\simeq 0$, hence (45) is satisfied. If $\tilde{Y}$ is a weakly exponential river of class S^1, the equation (45) is also satisfied since, when transformed by the telescope $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$, it expresses near-differentiability in 0, i.e.,

$$\frac{\tilde{y}_\omega(\gamma_\omega) - \tilde{y}_\omega(0)}{\gamma_\omega} \simeq 0.$$

This means that (45) is satisfied if $F_2'(\omega, \tilde{Y}(\omega))$ is positive. But rivers need not be S-differentiable under telescopes, an example is given by equation (17) with a standard and $0 < a < 1$. We have $\gamma_\omega = 1/\omega^a \simeq 0$, while $\tilde{Y}(X) = 1 + (-1)^X/X^a$ satisfies

$$\frac{\tilde{y}_\omega(\gamma_\omega) - \tilde{y}_\omega(0)}{\gamma_\omega} = (-1)^{\omega+1} \cdot 2.$$

With $a = 1$ we have thus an example of a polynomial attractive river which does not satisfy equation (45). Note however that if one admits two steps for $\tilde{Y}$ instead of one, equation (45) is satisfied, i.e., we always have

$$\frac{\tilde{Y}(\omega+2)-\tilde{Y}(\omega)}{\tilde{Y}(\omega)} = \varnothing \cdot (|F_2'(\omega, \tilde{Y}(\omega))| - 1). \tag{46}$$

In case $F_2'(\omega, \tilde{Y}(\omega)) > 0$, the formula follows from the fact that $\tilde{y}_\omega$ is of class S^1 under $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$, respectively $M_{\omega,\tilde{Y}(\omega)}$, and formula (5). In case $F_2'(\omega, \tilde{Y}(\omega)) < 0$, we use the equality

$$\frac{\tilde{Y}(\omega+2)-\tilde{Y}(\omega)}{\tilde{Y}(\omega)} = \frac{(\tilde{Y}(\omega+2)-\hat{Y}(\omega+2))-(\tilde{Y}(\omega)-\hat{Y}(\omega))}{\tilde{Y}(\omega)} + \frac{\hat{Y}(\omega+2)-\hat{Y}(\omega)}{\tilde{Y}(\omega)}. \tag{47}$$

Now $(\hat{Y}(\omega+2)-\hat{Y}(\omega))/\tilde{Y}(\omega) = \varnothing \cdot (|F_2'(\omega, \tilde{Y}(\omega))| - 1)$ because $\hat{y}_\omega$ is of class S^1 under $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$, respectively $M_{\omega,\tilde{Y}(\omega)}$, and by formula (6), and $((\tilde{Y}(\omega+2)-\hat{Y}(\omega+2))-(\tilde{Y}(\omega)-\hat{Y}(\omega)))/\tilde{Y}(\omega) = \varnothing \cdot (|F_2'(\omega, \tilde{Y}(\omega))| - 1)$ because $\tilde{y}_\omega - \hat{y}_\omega$ is of class $|S|^1$ under $T_{\omega,\tilde{Y}(\omega),1/\gamma_\omega}$, respectively $M_{\omega,\tilde{Y}(\omega)}$, and by formula (6) or formula (7).

The formulae (45) and (46) have obvious geometric interpretations. Let Φ be a solution such that $\Phi(\omega)/\tilde{Y}(\omega) \simeq 1$ and $\Phi(\omega) \neq \tilde{Y}(\omega)$. Put $\Delta = \Phi - \tilde{Y}$. Applying (19) and (1) we see that (45) expresses that

$$\frac{\tilde{Y}(\omega+1)-\tilde{Y}(\omega)}{\tilde{Y}(\omega)} = \varnothing \cdot \frac{|\Delta(\omega+1)| - |\Delta(\omega)|}{|\Delta(\omega)|},$$

i.e., slow evolution of the individual solution $\tilde{Y}$ with respect to the deviation of any solution within its asymptotic halo. In the nonalternating case verified for one step, by (46) it is true for all considered cases when we allow for two steps of the solution $\tilde{Y}$.

Acknowledgment I thank B.L.J Braaksma for stimulating discussions and several invitations to the University of Groningen.

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Verleger: Springer-Verlag GmbH, Sachsenplatz 4–6, 1201 Wien, Österreich. – Herausgeber: Dr. Nigel J. Cutland, Department of Mathematics, University of York, York, YO10 5DD, Großbritannien. – Redaktion: Sachsenplatz 4–6, 1201 Wien, Österreich. – Satz und Umbruch: Scientific Publishing Services (P) Ltd., Chennai, Indien. – Druck: Krips bv, Kaapweg 6, 7944 HV Meppel, Niederlande. – Verlagsort: Wien. – Herstellungsort: Meppel, Niederlande. – Printed in The Netherlands.
Offenlegung gem. § 25 Abs. 1 bis 3 Mediengesetz:
Unternehmensgegenstand: Verlag von wissenschaftlichen Büchern und Zeitschriften. An der Springer-Verlag GmbH ist beteiligt: Springer Austria GmbH, Sachsenplatz 4–6, 1201 Wien, Österreich, zu 99,8%. Geschäftsführer: Dr. Alexander Grossmann, Sachsenplatz 4–6, 1201 Wien, Österreich