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Compactness of ω^λ for λ singular

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Abstract: We characterize the compactness properties of the product of λ copies of the space ω with the discrete topology, dealing in particular with the case λ singular, using regular and uniform ultrafilters, infinitary languages and nonstandard elements. We also deal with products of uncountable regular cardinals with the order topology.

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1 Introduction

The problem of determining the compactness properties satisfied by powers of the countably infinite discrete topological space ω originates from Stone [19], who proved that ω^{ω_1} is not normal, hence, in particular, not Lindelöf. More generally, Mycielski [18] showed that ω^κ is not finally κ-compact, for every infinite cardinal κ strictly less than the first weakly inaccessible cardinal. Recall that a topological space is said to be *finally κ-compact* if any open cover has a subcover of cardinality strictly less than κ. *Lindelöfness* is the same as final ω_1-compactness. Previous work on the subject had been also done by A. Ehrenfeucht, P. Erdős, A. Hajnal and J. Łoś; see [18] for details. Mycielski's result cannot be generalized to arbitrarily large cardinals: if κ is weakly compact then ω^κ is indeed finally κ-compact: see Keisler and Tarski [10, Theorem 4.32]. Related work is due to D. V. Čudnovskiĭ, W. Hanf, D. Monk, D. Scott, S. Todorčević and S. Ulam, among many others.

With regard to powers of ω a more refined result has been obtained by Mrówka who, eg, in [17] showed that if the infinitary language $\mathcal{L}_{\omega_1,\omega}$ is (κ, κ)-compact then ω^κ is finally κ-compact. This is a stronger result since Boos [3] showed that it is possible that $\mathcal{L}_{\omega_1,\omega}$ is (κ, κ)-compact, even, that $\mathcal{L}_{\kappa,\omega}$ is (κ, κ)-compact, without κ being weakly compact. Recall that the *infinitary language* $\mathcal{L}_{\kappa,\omega}$ is like first-order logic, except that one allows

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conjunctions and disjunctions of $< \kappa$ many formulas. A set Γ of sentences of $\mathcal{L}_{\kappa,\omega}$ is λ-*satisfiable* if every $\Gamma' \subseteq \Gamma$ of cardinality $< \lambda$ is satisfiable. $\mathcal{L}_{\kappa,\omega}$ is (λ, λ)-*compact* if every λ-satisfiable set of λ sentences of $\mathcal{L}_{\kappa,\omega}$ is satisfiable. A cardinal κ is *weakly compact* if it is strongly inaccessible and $\mathcal{L}_{\kappa,\omega}$ is (κ, κ)-compact. Notice that, by Boos' result, without the assumption of strong inaccessibility we get non equivalent notions, in general. Asking only for (κ, κ)-compactness of $\mathcal{L}_{\omega_1,\omega}$ gives an even weaker notion, studied by some authors. See Bell [2]. See the result by A. Mekler stated in Eklof [8, Theorem 1.6] for another application of compactness of infinitary languages without the assumption of strong inaccessibility.

We now return to the specific aims of the present note. To the best of our knowledge, the gap between the results by Mycielski and Mrówka mentioned above had not been exactly filled until we showed in [16] that Mrówka gives the optimal bound, that is, for κ regular, ω^κ is finally κ-compact if and only if $\mathcal{L}_{\omega_1,\omega}$ is (κ, κ)-compact. The aim of the present note is to show that the result holds also for a singular cardinal κ. More generally, we provide a similar characterization for those pairs of cardinals λ and κ such that ω^κ is $[\lambda, \lambda]$-compact (Theorem 2.5). Recall that a topological space X is $[\lambda, \lambda]$-*compact* if every open cover of X by λ sets has a subcover by less than λ sets.

In order to give the proofs we need to use uniform and regular ultrafilters, as well as nonstandard elements. In particular, we shall introduce some related principles which may have independent interest and which, in a sense, measure "how hard it is" to exclude the uniformity of some ultrafilter, on one hand (Definition 2.3), or to omit the existence of a nonstandard element in some elementary extension, on the other hand (Definition 4.1). In Theorem 4.2 we show that, under a natural cardinality assumption, the two principles turn out to be equivalent. A large part of our methods work for arbitrary regular cardinals in place of ω; in particular, at a certain point, we shall make good use of a notion whose importance has been hinted in Chang [5] and which we call here being "μ-nonstandard"; in the particular case $\mu = \omega$ we get back the classical notion. These techniques allow us to provide a characterization of the compactness properties of products of (possibly uncountable) regular cardinals with the order topology (Corollary 3.2). This seems to have some interest since, as far as we know, all previously known results of this kind have dealt with cardinals endowed with the discrete topology (of course, the two situations coincide in the case of ω). More generally, in Proposition 3.4 we find a condition under which $[\lambda, \lambda]$-compactness of some product of topological spaces implies $[\mu, \mu]$-compactness of at least one factor.

The paper is divided as follows. In Section 2 we establish notations, we introduce our main principle $(\lambda, \lambda) \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ and we state the more general form of our results about powers of ω. We also make a comparison with the simpler situation described in [16], where we considered only the case when λ is regular. Section 3 deals with topology only; we provide a characterization of the principle $(\lambda, \lambda) \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ and we present some applications to products of cardinals. Only the first part of Section 3 shall be needed in the rest of the paper. In Section 4 we introduce the notion of a μ-nonstandard element and use it in order to provide a model-theoretical characterization of $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$, the particular case of the above principle when all the μ_γ's are equal to μ. We then have all the necessary elements to conclude the proof of Theorem 2.5, the main result on powers of ω.

2 A principle working for singular cardinals

Unexplained notions and notation are standard; see, eg, Chang and Keisler [6], Comfort and Negrepontis [7] and Jech [9]. Throughout, λ, μ and ν are infinite cardinals, κ is a (possibly finite) cardinal, α, β and γ are ordinals, X is a topological space and D is an ultrafilter. A cardinal μ is also considered as a topological space endowed either with the order topology or with the coarser *left order topology*. The latter topology consists of the intervals of the form $[0, \alpha)$ with $\alpha \leq \mu$. Notice that in some former works we have called the above topology the *initial intervals topology* however "left order topology" appears to be the standard terminology. No separation axioms are assumed throughout. Products of topological spaces are always assigned the Tychonoff topology. We need to introduce a finer notion of compactness for infinitary languages. If Σ and Γ are sets of sentences of $\mathcal{L}_{\omega_1,\omega}$ we say that Γ is *λ-satisfiable relative to* Σ if $\Sigma \cup \Gamma'$ is satisfiable, for every $\Gamma' \subseteq \Gamma$ of cardinality $< \lambda$. We say that $\mathcal{L}_{\omega_1,\omega}$ is *κ-(λ, λ)-compact* if $\Sigma \cup \Gamma$ is satisfiable, whenever $|\Sigma| \leq \kappa$, $|\Gamma| = \lambda$ and Γ is λ-satisfiable relative to Σ. We had formerly introduced the notion of κ-(λ, λ)-compactness (actually, a three cardinals version) for arbitrary logics, extending notions by H. J. Keisler, J. A. Makowsky, S. Shelah, A. Tarski, among others. See the book edited by Barwise and Feferman [1], Caicedo [4] and our [11] for more details and references. If $\kappa \leq \lambda$ then κ-(λ, λ)-compactness reduces to the classical notion of (λ, λ)-compactness recalled in the introduction. The main motivation behind the definition of κ-(λ, λ)-compactness is that sometimes we need to work only in models of some theory Σ; in fact usually Σ will be a fragment of the theory of some model. Considering only an appropriate fragment rather than the whole theory gives us the possibility of taking into account lower values of κ, thus providing much finer theorems. See Section 4. An ultrafilter D is *uniform over* μ if it is over μ and every member of D has cardinality μ. If D is an ultrafilter over some set I and $f\colon I \to J$ is a function, $f(D)$ is the ultrafilter over J defined by $Y \in f(D)$ if and only if $f^{-1}(Y) \in D$.

The key to our characterization of the compactness properties of ω^κ, as well as of other spaces, is a set theoretical principle which measures how hard it is to transfer regularity of ultrafilters from some cardinal λ to some cardinal μ. In the particular case $\mu = \omega$ it just measures how hard it is to show that certain ultrafilters are countably incomplete. The principle we shall use here and which allows the possibility of dealing with singular cardinals is technically involved. As a sort of an introduction to it, we first recall the relatively simpler principle used in Lipparini [16], which is suitable for dealing with regular cardinals.

Definition 2.1 We denote by $\lambda \not\Rightarrow (\mu_\gamma)_{\gamma\in\kappa}$ the following statement.

(*) For every sequence $(f_\gamma)_{\gamma\in\kappa}$ of functions such that $f_\gamma \colon \lambda \to \mu_\gamma$ for $\gamma \in \kappa$, there is some uniform ultrafilter D over λ such that, for no $\gamma \in \kappa$, $f_\gamma(D)$ is uniform over μ_γ.

We write $\lambda \not\Rightarrow^\kappa \mu$ when all the μ_γ's in (*) are equal to μ.

The negation of $\lambda \not\Rightarrow^\kappa \mu$ is denoted by $\lambda \Rightarrow^\kappa \mu$. A similar convention applies for $\lambda \Rightarrow (\mu_\gamma)_{\gamma\in\kappa}$

Notice that $\lambda \not\Rightarrow^\kappa \omega$ if and only if for every sequence of functions $f_\gamma \colon \lambda \to \omega$ $(\gamma \in \kappa)$ there is some uniform ultrafilter D over λ such that for every $\gamma \in \kappa$ the ultrafilter $f_\gamma(D)$ over ω is principal (since an ultrafilter over ω is uniform if and only if it is non principal.)

We notice that a reader interested only in compactness properties of powers of ω can limit herself or himself to the consideration of the principle $\lambda \not\Rightarrow^\kappa \omega$, and to the corresponding particular case $(\lambda, \lambda) \not\Rightarrow^\kappa \omega$ of the principle we shall introduce in Definition 2.3.

Notice also the similarity between the condition $\lambda \in \boldsymbol{M}$ studied by Mrówka (see, eg, [17, p. 704]) and the principle $\lambda \Rightarrow^\lambda \omega$. Indeed, had we changed the first occurrence of the word "uniform" in (*) in 2.1 to "non principal", the corresponding definition of $\lambda \Rightarrow^\lambda \omega$ would have turned exactly equivalent to $\lambda \in \boldsymbol{M}$ as defined in [17] (in both cases, the conditions are equivalent to the existence of λ partitions with suitable properties). Starting with uniform ultrafilters rather than with non principal ones is the key for making the principle work in the study of final λ-compactness of ω^λ. Moreover, our definition is more general, since it considers the possibility of having more than λ functions. This has applications to the study of the compactness properties of ω^κ. Furthermore, by allowing $\mu > \omega$ in $\lambda \not\Rightarrow^\kappa \mu$ we obtain theorems about compactness of powers of μ as a topological space with the order topology. In the case when λ is regular this has been done in Lipparini [16], where we have proved, among other things, the following theorem. Recall from the introduction the definition of $[\lambda, \lambda]$-compactness of a topological space.

Theorem 2.2 *If $\kappa \geq \lambda$ and λ is regular, the following statements are equivalent.*

(1) ω^κ *is* $[\lambda, \lambda]$*-compact.*

(2) $\mathcal{L}_{\omega_1,\omega}$ *is* κ*-*(λ, λ)*-compact.*

(3) $\lambda \not\Rightarrow^\kappa \omega$.

If λ is regular, then ω^λ is finally λ-compact if and only if $\mathcal{L}_{\omega_1,\omega}$ is (λ, λ)-compact.

In order to generalize the above theorem (as well as other results proved in [16]) to the case when λ is singular we need a variant of the principle introduced in Definition 2.1. The modified principle deals with regularity of ultrafilters rather than with uniformity. As usual, $[\lambda]^{<\lambda}$ denotes the set of all subsets of λ of cardinality $< \lambda$. We say that an ultrafilter D over $[\lambda]^{<\lambda}$ *covers* λ if $\{s \in [\lambda]^{<\lambda} \mid \alpha \in s\} \in D$, for every $\alpha \in \lambda$. The above definition is connected with the notion of a (λ, λ)-regular ultrafilter. Indeed, ultrafilters over $[\lambda]^{<\lambda}$ covering λ can be seen as standard witnesses of (λ, λ)-regularity: an ultrafilter D' over I is (λ, λ)*-regular* if there is a function $f \colon I \to [\lambda]^{<\lambda}$ such that $f(D')$ covers λ. This is one among many possible equivalent definitions of (λ, λ)-regularity. See Lipparini [13] for more details and for a comprehensive survey of results about regularity.

Definition 2.3 We denote by $(\lambda, \lambda) \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ the following statement.

(**) For every sequence of functions $(f_\gamma)_{\gamma \in \kappa}$ such that $f_\gamma \colon [\lambda]^{<\lambda} \to \mu_\gamma$ for $\gamma \in \kappa$, there is some ultrafilter D over $[\lambda]^{<\lambda}$ covering λ such that for no $\gamma \in \kappa$ the ultrafilter $f_\gamma(D)$ is uniform over μ_γ.

We write $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ when all the μ_γ's in (**) are equal to μ.

The negation of $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ is denoted by $(\lambda, \lambda) \Rightarrow^\kappa \mu$. A similar convention applies for $(\lambda, \lambda) \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$

Notice that $(\lambda, \lambda) \not\Rightarrow^\kappa \omega$ if and only if for every sequence of functions $f_\gamma \colon [\lambda]^{<\lambda} \to \omega$ $(\gamma \in \kappa)$ there is an ultrafilter D over $[\lambda]^{<\lambda}$ covering λ such that for every $\gamma \in \kappa$ the ultrafilter $f_\gamma(D)$ over ω is principal.

Only the particular case $(\lambda, \lambda) \not\Rightarrow^\kappa \omega$ will be involved in our study of compactness of ω^κ. However, when possible, we shall present the more general forms of our results (see

Theorem 4.2, or the whole of Section 3) since they appear to have intrinsic interest, and since proofs essentially use no additional argument. Only the arguments in the proof of Theorem 2.5 at the end of Section 4 apply only to the special case $\mu = \omega$. In the next proposition we show that the principles introduced in Definitions 2.1 and 2.3 are equivalent when λ is a regular cardinal. Moreover we show that if κ is a sufficiently large cardinal, then the principles $\lambda \not\Rightarrow^{\kappa} \mu$ and $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$ admit simple characterizations independent from κ. Recall that an ultrafilter D over I is *μ-decomposable* if there is $f\colon I \to \mu$ such that $f(D)$ is uniform over μ. Thus the conclusions in (*) and (**) of Definitions 2.1 and 2.3, respectively, assert the existence of some D such that no f_γ witnesses the μ_γ-decomposability of D (no matter whether D is actually μ_γ-decomposable or not).

Proposition 2.4 (1) $(\lambda, \lambda) \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ *implies* $\operatorname{cf} \lambda \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$

(2) $\operatorname{cf} \lambda \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ *implies* $\lambda \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$

(3) *If λ is regular then $\lambda \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$ if and only if $(\lambda, \lambda) \not\Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$*

(4) *If there is an ultrafilter uniform over λ which is not μ-decomposable then $\lambda \not\Rightarrow^{\kappa} \mu$. If $\kappa \geq \mu^{\lambda}$ then the converse holds, too.*

(5) *If there is a (λ, λ)-regular ultrafilter which is not μ-decomposable then $(\lambda, \lambda) \not\Rightarrow^{\kappa} \mu$. If $\kappa \geq \mu^{\lambda^{<\lambda}}$ then the converse holds, too.*

Proof (1) Fix an increasing sequence $(\lambda_\alpha)_{\alpha \in \operatorname{cf} \lambda}$ cofinal in λ. Define $g\colon \operatorname{cf} \lambda \to [\lambda]^{<\lambda}$ by $g(\alpha) = \lambda_\alpha$. If D' is uniform over $\operatorname{cf} \lambda$, then $g(D')$ over $[\lambda]^{<\lambda}$ covers λ. Thus if $f_\gamma\colon [\lambda]^{<\lambda} \to \mu_\gamma$ are functions as given by $(\lambda, \lambda) \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$, then the functions $f_\gamma \circ g\colon \operatorname{cf} \lambda \to \mu_\gamma$ witness $\operatorname{cf} \lambda \Rightarrow (\mu_\gamma)_{\gamma \in \kappa}$

(2) Define $h\colon \lambda \to \operatorname{cf} \lambda$ by $h(\beta) = \inf\{\alpha < \lambda \mid \beta < \lambda_\alpha\}$. If D' is uniform over λ, then $h(D')$ is uniform over $\operatorname{cf} \lambda$, and we can argue as before.

In other words, we have proved, for a natural extension of our notation, $\operatorname{cf} \lambda \Rightarrow^{1} (\lambda, \lambda)$, as well as $\lambda \Rightarrow^{1} \operatorname{cf} \lambda$. Clauses (1) and (2) follow from the above relations by an obvious transitivity property of our double-arrow principles. Compare Lipparini [11, Lemma 0.15(i)] and [14, Proposition 6.5]. Let us remark that in [11] the parameter κ does not appear explicitly: the notation there parallels the present one when κ is assumed to be at least as large as all the cardinals involved: see the remark before Theorem 2.5 here.

(3) It is enough to prove the equivalence of the negations of the principles. An implication follows from (1). For the reverse implication, by the mentioned transitivity property of the double-arrow relation, it is enough to show $(\lambda, \lambda) \Rightarrow^{1} \lambda$, for λ regular.

This is witnessed by the function $k\colon [\lambda]^{<\lambda} \to \lambda$ defined by $k(x) = \sup x$, for $x \in [\lambda]^{<\lambda}$ (the range of k is contained in λ since λ is regular).

(4) The first statement follows trivially from the definitions. The second statement follows from the fact that there are μ^λ functions from λ to μ.

(5) Suppose that there is an ultrafilter D' which is (λ, λ)-regular and not μ-decomposable. Say, D' is over I. Since D' is (λ, λ)-regular, then, by definition, there is a function $g\colon I \to [\lambda]^{<\lambda}$ such that $D = g(D')$ is over $[\lambda]^{<\lambda}$ and covers λ. It is standard to see that D is not μ-decomposable, since D' is not μ-decomposable (see, eg, [13, Properties 1.1(x)]). Thus $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$. For the converse, since there are $\mu^{\lambda^{<\lambda}}$ functions from $[\lambda]^{<\lambda}$ to μ, by applying $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ to the family of such functions, we get an ultrafilter D over $[\lambda]^{<\lambda}$ which covers λ and which is not μ-decomposable. Now observe that D is (λ, λ)-regular, by definition. □

Proposition 2.4(4)(5) shows that the principles $\lambda \not\Rightarrow^\kappa \mu$ and $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ are interesting only for small values of κ. While in general we obtain a stronger statement when we increase κ in the above principles, at the points $\kappa = \mu^\lambda$ and $\kappa = \mu^{\lambda^{<\lambda}}$, respectively, we have already reached the strongest possible notion. The problem of the existence of ultrafilters as in 2.4(4)(5), for various λ and μ, is connected with difficult set-theoretical problems involving large cardinals, forcing and pcf-theory, and has been widely studied, sometimes in equivalent formulations. See Lipparini [13] for more information. In a couple of papers we have somewhat attempted a study of the more comprehensive (hence more difficult!) relations $\lambda \not\Rightarrow^\kappa \mu$ and $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$. See [11] and some references there. Roughly, while, on one hand, for large κ we get notions related to measurability, on the other hand, for smaller values of κ we get corresponding variants of weak compactness, as the present note itself exemplifies.

Remark Notice that in some previous works we had given the definition of, say, $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ by means of the equivalent condition that we are going to present in Theorem 4.2 below. The equivalence holds only under the assumption $\kappa \geq \sup\{\lambda, \mu\}$. Hence the notation we have previously used in some places might be not consistent with the present one (but only when small values of κ are taken into account).

Notice also that, as we mentioned in the proof of Proposition 2.4, we can naturally extend the present notation in order to consider principles like, say, $\lambda \not\Rightarrow^\kappa (\mu, \mu)$. The arguments in the proof of 2.4(1)-(3) show that if μ is a regular cardinal, then $\lambda \not\Rightarrow^\kappa (\mu, \mu)$ is equivalent to $\lambda \not\Rightarrow^\kappa \mu$. This equivalence should be taken into account when comparing the present results and terminology with some previous ones.

Theorem 2.5 *If $\kappa \geq \lambda$ then the following statements are equivalent.*

(1) ω^κ *is* $[\lambda, \lambda]$*-compact.*

(2) $\mathcal{L}_{\omega_1,\omega}$ *is* κ*-*(λ, λ)*-compact.*

(3) $(\lambda, \lambda) \not\Rightarrow^\kappa \omega$.

The proof of Theorem 2.5 shall be deferred until the end of Section 4, in order to state and prove preliminary results.

Corollary 2.6 *The following statements are equivalent.*

(1) ω^λ *is finally* λ*-compact.*

(2) $\mathcal{L}_{\omega_1,\omega}$ *is* (λ, λ)*-compact.*

(3) $(\lambda, \lambda) \not\Rightarrow^\lambda \omega$.

Proof It is easy to show that a topological space is finally λ-compact if and only if it is both $[\lambda, \lambda]$-compact and finally λ^+-compact. Notice that ω^λ is finally λ^+-compact, since it has a base of cardinality λ. Hence ω^λ is finally λ-compact if and only if it is $[\lambda, \lambda]$-compact. Thus the corollary is the particular case $\kappa = \lambda$ of Theorem 2.5. $\square$

By Proposition 2.4(3), if λ is a regular cardinal, then Theorem 2.5 and Corollary 2.6 give essentially the same result as Theorem 2.2.

3 Topological equivalents

If D is an ultrafilter over some set I, a point $x \in X$ is said to be a *D-limit point* of a sequence $(x_i)_{i \in I}$ of elements of X if $\{i \in I \mid x_i \in U\} \in D$, for every open neighborhood U of x. To avoid complex expressions in subscripts, we sometimes shall denote a sequence $(x_i)_{i \in I}$ as $\langle x_i \mid i \in I \rangle$. The next theorem follows easily from Caicedo [4, Section 3], which extended, generalized and simplified former results by A. R. Bernstein, J. Ginsburg and V. Saks, among others. A detailed proof in an even more general context can be found in Lipparini [15, Theorem 2.3], taking $\lambda = 1$ there. See the last paragraph on [15, p. 2509].

Theorem 3.1 *A topological space X is $[\lambda, \lambda]$-compact if and only if for every sequence $\langle x_s \mid s \in [\lambda]^{<\lambda} \rangle$ of elements of X there exists some ultrafilter D over $[\lambda]^{<\lambda}$ such that D covers λ and $\langle x_s \mid s \in [\lambda]^{<\lambda} \rangle$ has some D-limit point in X.*

Corollary 3.2 *Suppose that $(\mu_\gamma)_{\gamma\in\kappa}$ is a sequence of regular cardinals and that each μ_γ is endowed either with the order topology or with the left order topology. Then $\prod_{\gamma\in\kappa}\mu_\gamma$ is $[\lambda,\lambda]$-compact if and only if $(\lambda,\lambda)\not\Rightarrow(\mu_\gamma)_{\gamma\in\kappa}$*

Proof Let $X=\prod_{\gamma\in\kappa}\mu_\gamma$ and, for $\gamma\in\kappa$, let $\pi_\gamma\colon X\to\mu_\gamma$ be the natural projection. A sequence of functions as in the first line of (**) in Definition 2.3 can be naturally identified with a sequence $\langle x_s \mid s\in[\lambda]^{<\lambda}\rangle$ of elements of X, by posing $\pi_\gamma(x_s)=f_\gamma(s)$. By Theorem 3.1, X is $[\lambda,\lambda]$-compact if and only if, for every sequence $\langle x_s \mid s\in[\lambda]^{<\lambda}\rangle$ of elements of X, there is an ultrafilter D over $[\lambda]^{<\lambda}$ covering λ and such that $\langle x_s \mid s\in[\lambda]^{<\lambda}\rangle$ has a D-limit point in X. Since a sequence in a product has a D-limit point if and only if each component has a D-limit point, the above condition holds if and only if, for each $\gamma\in\kappa$, $\langle\pi_\gamma(x_s)\mid s\in[\lambda]^{<\lambda}\rangle$ has a D-limit point in μ_γ. This happens if and only if, for each $\gamma\in\kappa$, there is $\delta_\gamma\in\mu_\gamma$ such that $\{s\in[\lambda]^{<\lambda}\mid \pi_\gamma(x_s)<\delta_\gamma\}\in D$, no matter whether μ_γ has the left order or the order topology. Under the mentioned identification, and since every μ_γ is regular, this means exactly that each $f_\gamma(D)$ fails to be uniform over μ_γ. The assumption that μ_γ is regular is necessary, since if μ_γ is singular, it could happen that $f_\gamma(D)$ concentrates on a set of cardinality $<\mu_\gamma$, yet $\{f_\gamma(s)\mid s\in[\lambda]^{<\lambda}\}$ is unbounded mod D in μ_γ (thus has no D-limit point in μ_γ). □

We end the present section by showing that, though we have stated our topological results in terms of products of cardinals, they can be reformulated in a way that involves arbitrary products of topological spaces. The remaining part of this section shall not be used in the rest of the paper; in particular, it will not be used in the proof of Theorem 2.5, so the reader might decide to skip it.

Lemma 3.3 *If μ is a regular cardinal and X is a topological space, then X is not $[\mu,\mu]$-compact if and only if there is a continuous surjective function $f\colon X\to\mu$, where μ is endowed with the left order topology.*

Proof One implication is trivial, since μ is not $[\mu,\mu]$-compact, μ being a regular cardinal, and since $[\mu,\mu]$-compactness is preserved under continuous surjective images.

For the reverse implication, it is well-known that if μ is a regular cardinal, then a topological space X is not $[\mu,\mu]$-compact if and only if there is a decreasing sequence $(C_\alpha)_{\alpha\in\mu}$ of nonempty closed subsets of X with empty intersection. See, eg, Lipparini [14, Theorem 4.4] for the proof in a more general context. Without loss of generality, we can assume that $C_0=X$ and that $(C_\alpha)_{\alpha\in\mu}$ is strictly decreasing, since μ

is regular. Define $f\colon X \to \mu$ by $f(x) = \sup\{\alpha \in \mu \mid x \in C_\alpha\}$. Notice that the range of f is contained in μ, since the sequence $(C_\alpha)_{\alpha\in\mu}$ is decreasing with empty intersection. Moreover, f is continuous, since $f^{-1}([\beta,\mu)) = C_\beta$, if $\beta \in \mu$ is a successor ordinal, and $f^{-1}([\beta,\mu)) = \bigcap_{\alpha<\beta} C_\alpha$, if $\beta \in \mu$ is limit; hence the counterimage by f of a closed set is closed. Since $C_0 = X$ and $(C_\alpha)_{\alpha\in\mu}$ is strictly decreasing, then f is surjective. □

Proposition 3.4 *If $(\mu_\gamma)_{\gamma\in\kappa}$ is a sequence of regular cardinals then the following statements are equivalent.*

(1) $\prod_{\gamma\in\kappa} \mu_\gamma$ *is not $[\lambda,\lambda]$-compact, where each μ_γ is equivalently endowed either with the order topology or with the left order topology.*

(2) $(\lambda,\lambda) \Rightarrow (\mu_\gamma)_{\gamma\in\kappa}$

(3) *For every set I and every product $X = \prod_{i\in I} X_i$ of topological spaces, if X is $[\lambda,\lambda]$-compact, then for every injective function $g\colon \kappa \to I$ there is $\gamma \in \kappa$ such that $X_{g(\gamma)}$ is $[\mu_\gamma,\mu_\gamma]$-compact.*

(4) *For every product $X = \prod_{\gamma\in\kappa} X_\gamma$ of topological spaces, if X is $[\lambda,\lambda]$-compact, then there is $\gamma \in \kappa$ such that X_γ is $[\mu_\gamma,\mu_\gamma]$-compact.*

Proof (1) $\Leftrightarrow$ (2) is Corollary 3.2 in contrapositive form.

(3) $\Rightarrow$ (4) is trivial by taking $I = \kappa$ and g the identity function.

(4) $\Rightarrow$ (1) is trivial, observing that μ_γ is not $[\mu_\gamma,\mu_\gamma]$-compact (with either topology), since μ_γ is regular.

To finish the proof we shall prove that if (3) fails then (1) fails. Suppose that (3) fails, as witnessed by some $[\lambda,\lambda]$-compact $X = \prod_{i\in I} X_i$ and an injective $g\colon \kappa \to I$ such that no $X_{g(\gamma)}$ is $[\mu_\gamma,\mu_\gamma]$-compact. By Lemma 3.3 for each $\gamma \in \kappa$ we have a continuous surjective function $h_\gamma\colon X_\gamma \to \mu_\gamma$ and, by naturality of products and since g is injective, a continuous surjective $h\colon X \to \prod_{\gamma\in\kappa} \mu_\gamma$. Thus if X is $[\lambda,\lambda]$-compact then so is $\prod_{\gamma\in\kappa} \mu_\gamma$, since $[\lambda,\lambda]$-compactness is preserved under surjective continuous images, hence (1) fails in the case the μ_γ's are assigned the left order topology. This is enough, since we have already proved that in (1) we can equivalently consider either topology, since in each case (1) is equivalent to (2). □

Notice that, in particular, it follows from Proposition 3.4 that if μ is regular, $(\lambda,\lambda) \Rightarrow^\kappa \mu$ and some product is $[\lambda,\lambda]$-compact, then all but at most $<\kappa$ factors are $[\mu,\mu]$-compact. In this way we obtain alternative proofs— as well as various strengthenings— of many of the results we have proved in [12]. Notice also that Proposition 3.4 shows that the assumption that the spaces under consideration are T_1 is unnecessary in Lipparini [16, Proposition 9].

4 A further equivalence in terms of nonstandard elements

Definition 4.1 We now need to consider a model $\mathfrak{A}(\lambda, \mu)$ which contains both a copy of $\langle [\lambda]^{<\lambda}, \subseteq, \{\alpha\} \rangle_{\alpha \in \lambda}$ and a copy of $\langle \mu, <, \beta \rangle_{\beta \in \mu}$, where the $\{\alpha\}$'s and the β's are interpreted as constants. Though probably the most elegant way to accomplish this is by means of a two-sorted model, we do not want to introduce technicalities and simply assume that $A = [\lambda]^{<\lambda} \cup \mu$ and that $[\lambda]^{<\lambda}$ and μ are interpreted, respectively, by unary predicates U and V. In details, we let $\mathfrak{A}(\lambda, \mu) = \langle A, U, V, \subseteq, <, \{\alpha\}, \beta \rangle_{\alpha \in \lambda, \beta \in \mu}$, where $U(s)$ holds in $\mathfrak{A}(\lambda, \mu)$ if and only if $s \in [\lambda]^{<\lambda}$ and $V(c)$ holds in $\mathfrak{A}(\lambda, \mu)$ if and only if $c \in \mu$. By abuse of notation we shall not distinguish between symbols and their interpretations. If $\mathfrak{A}$ is an expansion of $\mathfrak{A}(\lambda, \mu)$ and $\mathfrak{B} \equiv \mathfrak{A}$ (that is, $\mathfrak{B}$ is *elementarily equivalent* to $\mathfrak{A}$), we say that $b \in B$ *covers* λ if $U(b)$ and $\{\alpha\} \subseteq b$ hold in $\mathfrak{B}$, for every $\alpha \in \lambda$. We say that $c \in B$ is *μ-nonstandard* if $V(c)$ and $\beta < c$ hold in $\mathfrak{B}$, for every $\beta \in \mu$. Of course, in the case $\mu = \omega$, we get the usual notion of a nonstandard element. Notice that if D is an ultrafilter over $[\lambda]^{<\lambda}$ then D covers λ in the ultrafilter sense (cf. the paragraph immediately before Definition 2.3) if and only if the D-class $[Id]_D$ of the identity on $[\lambda]^{<\lambda}$ in the ultrapower $\prod_D \mathfrak{A}(\lambda, \mu)$ covers λ in the present sense. Moreover, if μ is regular, then an ultrafilter D over μ is uniform if and only if $\prod_D \mathfrak{A}(\lambda, \mu)$ has a μ-nonstandard element.

Theorem 4.2 *If $\kappa \geq \sup\{\lambda, \mu\}$ then $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ if and only if for every expansion $\mathfrak{A}$ of $\mathfrak{A}(\lambda, \mu)$ with at most κ new symbols (equivalently, symbols and sorts), there is $\mathfrak{B} \equiv \mathfrak{A}$ such that $\mathfrak{B}$ has an element covering λ but no μ-nonstandard element.*

Proof Suppose that $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ and let $\mathfrak{A}$ be an expansion of $\mathfrak{A}(\lambda, \mu)$ with at most κ new symbols and sorts. Without loss of generality we may assume that $\mathfrak{A}$ has Skolem functions, since this adds at most $\kappa \geq \sup\{\lambda, \mu\}$ new symbols. Enumerate as $(f_\gamma)_{\gamma \in \kappa}$ all the functions from $[\lambda]^{<\lambda}$ to μ which are definable in $\mathfrak{A}$ (repeat occurrences, if necessary), and let D be the ultrafilter given by $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$. Let $\mathfrak{C}$ be the ultrapower $\prod_D \mathfrak{A}$. By the remark before the statement of the theorem, $b = [Id]_D$ is an element in $\mathfrak{C}$ which covers λ. Let $\mathfrak{B}$ be the Skolem hull of $\{b\}$ in $\mathfrak{C}$; thus $\mathfrak{B} \equiv \mathfrak{C} = \prod_D \mathfrak{A} \equiv \mathfrak{A}$, and b covers λ in $\mathfrak{B}$. If by contradiction $\mathfrak{B}$ has a μ-nonstandard element c, then there is some $\gamma \in \kappa$ such that $c = f_\gamma(b)$, by the definition of $\mathfrak{B}$. Thus $c = f_\gamma([Id]_D) = [f_\gamma]_D$, but, since μ is regular, this implies that $f_\gamma(D)$ is uniform over μ, contradicting the choice of D.

For the converse, suppose that $(f_\gamma)_{\gamma \in \kappa}$ is a sequence of functions from $[\lambda]^{<\lambda}$ to μ. Let $\mathfrak{A}$ be the expansion of $\mathfrak{A}(\lambda, \mu)$ obtained by adding the f_γ's as unary functions.

Notice that we have no need to introduce new sorts. By assumption, there is $\mathfrak{B} \equiv \mathfrak{A}$ with an element b covering λ but without μ-nonstandard elements. For every formula $\varphi(y)$ in the vocabulary of $\mathfrak{A}$ and with exactly one free variable y, let $Z_\varphi = \{s \in [\lambda]^{<\lambda} \mid \varphi(s) \text{ holds in } \mathfrak{A}\}$. Put $E = \{Z_\varphi \mid \varphi \text{ is as above and } \varphi(b) \text{ holds in } \mathfrak{B}\}$. E has trivially the finite intersection property, thus it can be extended to some ultrafilter D over $[\lambda]^{<\lambda}$. For each $\alpha \in \lambda$, considering the formula $\varphi_\alpha\colon \{\alpha\} \subseteq y$, we get that $Z_{\varphi_\alpha} = \{s \in [\lambda]^{<\lambda} \mid \alpha \in s\} \in E \subseteq D$, thus D covers λ. Let $\gamma \in \kappa$. Since $\mathfrak{B}$ has no μ-nonstandard element, there is $\beta < \mu$ such that $f_\gamma(b) < \beta$ holds in $\mathfrak{B}$. Letting $\varphi_\gamma(y)$ be $f_\gamma(y) < \beta$, we get that $Z_{\varphi_\gamma} = \{s \in [\lambda]^{<\lambda} \mid f_\gamma(s) < \beta\} \in E \subseteq D$, thus $[0, \beta) \in f_\gamma(D)$, proving that $f_\gamma(D)$ is not uniform over μ. □

Theorem 4.2 explains the reason why we have used a negated implication sign in the notation $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$. The principle asserts that, modulo possible expansions, the existence of an element covering λ does *not* necessarily *imply* the existence of a μ-nonstandard element. Similarly, $(\lambda, \lambda) \not\Rightarrow^\kappa \mu$ is equivalent to the statement that $[\lambda, \lambda]$-compactness of a product of κ-many topological spaces does not imply the $[\mu, \mu]$-compactness of some factor. See Proposition 3.4.

Proof of Theorem 2.5 The equivalence of (1) and (3) is the particular case of Corollary 3.2 when all the μ_γ's are equal to ω. Thus, in view of Theorem 4.2, and since $\kappa \geq \lambda$, it is enough to prove that (2) is equivalent to the necessary and sufficient condition given in 4.2 for $(\lambda, \lambda) \not\Rightarrow^\kappa \omega$. For the simpler direction, suppose that $\mathcal{L}_{\omega_1,\omega}$ is κ-(λ, λ)-compact and that $\mathfrak{A}$ is an expansion of $\mathfrak{A}(\lambda, \omega)$ with at most κ new symbols. Let Σ be the elementary (first order) theory of $\mathfrak{A}$ plus an $\mathcal{L}_{\omega_1,\omega}$ sentence asserting that there exists no nonstandard element. Let $\Gamma = \{\{\alpha\} \subseteq b \mid \alpha \in \lambda\}$. By applying the κ-(λ, λ)-compactness of $\mathcal{L}_{\omega_1,\omega}$ to the above sets of sentences we get a model $\mathfrak{B}$ as requested by the condition in Theorem 4.2.

The reverse direction is a variation on a standard reduction argument. Suppose that the condition in Theorem 4.2 holds. If $\mathfrak{A}$ is a many-sorted expansion of the model $\mathfrak{A}(\lambda, \omega)$ introduced in Definition 4.1 then, for every $\mathfrak{B} \equiv \mathfrak{A}$ such that $\mathfrak{B}$ has no nonstandard element, a formula ψ of $\mathcal{L}_{\omega_1,\omega}$ of the form $\bigwedge_{n\in\omega} \varphi_n(\bar{x})$ is equivalent to $\forall y(V(y) \Rightarrow R_\psi(y, \bar{x}))$ in some expansion $\mathfrak{B}^+$ of $\mathfrak{B}$ with a newly introduced relation R_ψ such that $\forall \bar{x}(R_\psi(n, \bar{x}) \Leftrightarrow \varphi_n(\bar{x}))$ holds in $\mathfrak{B}^+$, for every $n \in \omega$. Here we are using in an essential way the fact that in a sentence of $\mathcal{L}_{\omega_1,\omega}$ we can quantify away only a finite number of variables, hence we can do with a finitary relation R_ψ. Thus, given Σ and Γ sets of sentences as in the definition of κ-(λ, λ)-compactness, assuming that we work in models without nonstandard elements, iterating the above procedure for all subformulas of the sentences under consideration and working in some appropriately expanded vocabulary, we may reduce all the relevant satisfaction conditions to the case in which Σ and Γ are sets of first order sentences. In the situation at hand we need to add to Σ all the sentences of the form $\forall \bar{x}(R_\psi(n,\bar{x}) \Leftrightarrow \varphi_n(\bar{x}))$ as above, but easy computations show that we still have $|\Sigma| \leq \kappa$, since $\kappa \geq \lambda$. If $\Gamma = \{\gamma_\alpha \mid \alpha \in \lambda\}$ is λ-satisfiable relative to Σ, construct a many-sorted expansion $\mathfrak{A}$ of $\mathfrak{A}(\lambda,\omega)$ with a further new binary relation S such that, for every $s \in [\lambda]^{<\lambda}$, $\{z \in A \mid S(s,z)\}$ models $\Sigma \cup \{\gamma_\alpha \mid \alpha \in s\}$. This is possible, since Γ is λ-satisfiable relative to Σ. We claim that if $\mathfrak{B} \equiv \mathfrak{A}$ is given by the equivalent condition for $(\lambda,\lambda) \not\Rightarrow^\kappa \omega$ in Theorem 4.2 and $b \in B$ covers λ, then $\{z \in B \mid S(b,z)\}$ models $\Sigma \cup \Gamma$. Indeed, for every $\alpha \in \lambda$ the sentence $\forall w(\{\alpha\} \subseteq w \wedge U(w) \Rightarrow \gamma_\alpha^{S(w,-)})$ is satisfied in $\mathfrak{A}$ hence, by elementarity, it is satisfied in $\mathfrak{B}$, too. Here $\gamma_\alpha^{S(w,-)}$ denotes a *relativization* of γ_α to $S(w,-)$, that is, a sentence such that if $\mathfrak{C}$ is a model, $c \in C$ and $\{z \in C \mid S(c,z)\}$ is itself (the base set of) a model for the appropriate vocabulary, then $\gamma_\alpha^{S(c,-)}$ is satisfied in $\mathfrak{C}$ if and only if γ_α is satisfied in $\{z \in C \mid S(c,z)\}$. Similarly, for every $\sigma \in \Sigma$, $\forall w(U(w) \Rightarrow \sigma^{S(w,-)})$ is satisfied in $\mathfrak{A}$ hence it is satisfied in $\mathfrak{B}$. This shows that $\{z \in B \mid S(b,z)\}$ models $\Sigma \cup \Gamma$. See the book edited by Barwise and Feferman [1] for further technical details, in particular about relativization and about dealing with constants. Notice also that in the above proof we do need the many-sorted (or relativized) version of the condition in Theorem 4.2, since when κ is large the models witnessing the λ-satisfiability of Γ relative to Σ might have cardinality exceeding the cardinality of the base set of $\mathfrak{A}(\lambda,\omega)$. $\square$

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