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A constructive version of the extremum value theorem for spaces of vector–valued functions

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Abstract: It is shown that the extremum value theorem for spaces of two–dimensional vector–valued functions in an approximate format admits a proof in the sense of Bishop's constructive mathematics. The proof is based on an explicit construction of functions that build an approximation to the original function space.

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1 Introduction

The classical extremum value theorem is stated as follows:

Theorem 1.1 *Let f be a continuous function on a compact interval $[a, b]$. Then, there exist real numbers $x, y \in [a, b]$ such that $f(x) = \sup f$ and $f(y) = \inf f$.*

This theorem can be generalized to compact metric spaces including function spaces. Consider, for example, the following optimization problem:

$$\begin{aligned} &\inf_{v} \qquad J[v] = \vartheta\left(x(t_1)\right) + \int_{t_0}^{t_1} \mathcal{L}\left(x(t), v(x(t)), t\right) \,\mathrm{d}t, \\ &\text{such that} \qquad \dot{x}(t) = f\left(x(t), v(x(t)), t\right), x(t_0) = x_0, \end{aligned} \tag{1}$$

where J is the cost functional, the variable t denotes time, x is called *state* and an element of some compact normed space X, also called *state space*, and function v is called *control law*. The function ϑ is called *endpoint penalty* and $\mathcal{L}$ is the *Lagrangian*, also called *running cost*. Such an optimization problem is, for example, addressed in optimal control. To solve (1) means to find a control law $v^*(x)$ such that $J[v^*] = \inf_v J[v]$. For example, in case of robot navigation on a plane, we might look for the optimal ground speed vector as a function of the robot position.

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From the standpoint of constructive mathematics by Bishop, Theorem 1.1 does not hold. Even though the numbers $\sup f$ and $\inf y$ may exist constructively, they might not be attained. This means that there is no way to construct the numbers x and y as in the statement of Theorem 1.1. In case of (1), the optimal control law may exist classically, but might not be computable.

In this work, we address this problem and prove the extremum value theorem in an approximate format constructively.[1] As the underlying metric space, we take spaces of functions from an arbitrarily dimensioned Euclidean space to the plane. The theorem is stated as follows:

Theorem 1.2 *Let $\mathcal{F}$ be the space of all uniformly Lipschitz and uniformly bounded functions from a closed hypercube $\bar{\mathcal{H}}(b, K) \subset \mathbb{R}^n$ to the plane $\mathbb{R}^2$, and let J be a uniformly continuous functional from $\mathcal{F}$ to $\mathbb{R}$. Then, for any $k \in \mathbb{N}$, there exists an $f^* \in \mathcal{F}$ such that $J[f^*] - \frac{1}{k} \leq \inf J$.*

The constructive notions used to prove this theorem are discussed in detail in Section 2. Relaxing the statement of the theorem and considering approximate extrema can be justified for practical applications such as planar robot navigation. The key aspect of the proof of the main result is showing that, under certain conditions, the function space in question admits a finite approximation.

2 Preliminaries

For the proof of Theorem 1.2, we require certain notions from constructive mathematics and basic lemmas. For a comprehensive description, the reader may refer to Bishop and Bridges [3], Bridges and Richman [5], Bridges and Vita [6], Ye [13] and Schwichtenberg [10]. First, we address the basics and recall briefly the notions of real numbers and sets in Section 2.1. After that, we discuss Euclidean spaces in Section 2.2. Finally, in Section 2.3, the constructive notions of functions and functionals are provided.

[1]A result extending [3, Theorem 5.6] for functions that map to R^n using the Axiom of Countable Choice was suggested by an anonymous referee. In the current work, the Axiom of Countable Choice is not required and constructions of approximating functions are done explicitly

2.1 Basics

Bishop's constructive mathematics uses the notion of an **operation** which is an algorithm that produces a unique result in a finite number of steps for each input in its domain. For example, a **real** number x is a *regular* Cauchy sequence of rational numbers in the sense that

$$\forall n, m \in \mathbb{N}. |x(n) - x(m)| \leq \frac{1}{n} + \frac{1}{m}$$

where $x(n)$ is an *operation* that produces the nth rational approximation to x. Notice that $\forall n. |x(n) - x| \leq \frac{2}{n}$ which means that a Cauchy sequence of rationals converges to the real it represents with $2n$ being the **modulus of convergence**. Convergence moduli for a sum and product of real numbers are easily defined. A Cauchy sequence of reals is also provided with a modulus of convergence.

A **set** is a pair of operations: $\in$ determines that a given object is a member of the set, and $=$ determines whenever two given set members are equal. Existence and universal quantifiers are interpreted as follows: $\exists x \in A. \varphi[x]$ means that an operation has been derived that constructs an instance x along with a proof of $x \in A$, and a proof of the logical formula $\varphi[x]$ as *witnesses*; $\forall x \in A. \varphi[x]$ means that an operation has been derived that proves $\varphi[x]$ for any x provided with a witness for $x \in A$. A set A is called **inhabited** if there exists an $x \in A$. A **finite set** is a set that admits a bijection to the set $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ which means that all its elements are enumerable.

2.2 Euclidean space

The Euclidean space $\mathbb{R}^n$ is a normed space with the norm $\|x\| \triangleq \left(\sum_{i=1}^{n} (x_i)^2\right)^{\frac{1}{2}}$ where x_i is the ith coordinate of x. The metric is defined as $\|x - y\|$ for any $x, y \in \mathbb{R}^n$. We will use the notions of a ball and hypercube in which we focus solely on rational centers, radii and side lengths. A (rational) closed **ball** $\bar{\mathcal{B}}(b, K)$ in $\mathbb{R}^n$ with a radius $K \in \mathbb{Q}$, $K > 0$ centered at $b \in \mathbb{Q}^n$ is the set $\{x : x \in \mathbb{R}^n \wedge \|x - b\| \leq K\}$. For a (rational) closed *hypercube* centered at $b \in \mathbb{Q}^n$ with a side length $2K \in \mathbb{Q}$, $K > 0$, we will use the notation $\bar{\mathcal{H}}(b, K)$.

A **regular mesh** on $\mathbb{R}^n$ with a step $\delta \in \mathbb{Q}$ is the set of points

$$\{(k_{1i}\delta, \dots, k_{ni}\delta) : \forall j = 1, \dots, n. k_{ji} \in \mathbb{Z}\}_i .$$

For example, a regular mesh on $\mathbb{R}$ with a step 1 is the set of points

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} .$$

We will also consider regular meshes on closed balls and hypercubes. Due to Lemma 4.1 from Beeson [2, page 8], we can decide whether $x = y$ or $x \neq y$ for arbitrary algebraic numbers x, y. Notice that a closed ball is given by an expression that contains a polynomial of several variables and a radical. This allows deciding whether a given point belongs to the given closed ball (or hypercube) provided that the point possesses solely rational (or, in general, algebraic) coordinates. Consequently, distances between closed balls, hypercubes, their unions and boundaries, are algebraic numbers since their computation involves only finite products, sums and radicals of algebraic numbers. Further, we may define a regular mesh on a closed ball, respectively hypercube, as a finite set that is the intersection of the ball, respectively hypercube, with the regular mesh on the entire Euclidean space. The following lemma helps locate an arbitrary point in a closed hypercube:

Lemma 2.1 *Let $P = \{p_i\}_i$ be a regular mesh on a closed hypercube $\bar{\mathcal{H}}(0, K) \subset \mathbb{R}^n$ with a step $\delta \leq K$, and $x \in \bar{\mathcal{H}}(0, K)$. Then, there exists a closed hypercube $\bar{\mathcal{H}}(p_i, \delta)$, $p_i \in P$ such that $x \in \bar{\mathcal{H}}(p_i, \delta)$.*

Proof Since x is a tuple of n real numbers $(x_1, \ldots, x_n)$, we may assume that mth rational approximation to x is a tuple of rational numbers $x(m) := (x_1(n \cdot m), \ldots, x_n(n \cdot m))$. Indeed, for any $m' \in \mathbb{N}$, $\|x(m) - x(m')\| \leq \sqrt{n} \max_i |x_i(n \cdot m) - x_i(n \cdot m')| \leq \sqrt{n}\left(\frac{1}{n \cdot m} + \frac{1}{n \cdot m'}\right) \leq \frac{1}{m} + \frac{1}{m'}$ since $|x_i(n \cdot m) - x_i(n \cdot m')| \leq \frac{1}{n \cdot m} + \frac{1}{n \cdot m'}$ for all $i \in \{1, \ldots, n\}$. Let $m := \lceil \frac{4}{\delta} \rceil$, then $\|x(m) - x\| \leq \frac{\delta}{2}$. Compute all distances $||x(m) - p_i||$, $p_i \in P$. Since they are algebraic numbers, we can decide whether $||x(m) - p_i|| < \frac{\delta}{2}$. If this is the case for some p_i, then $x \in \bar{\mathcal{H}}(p_i, \delta)$. If there are more than one such balls, we pick the one with the smallest index. If $||x(m) - p_j|| = \frac{\delta}{2}$ for some indices $j = j_1, \ldots, j_L$, then we pick the smallest such j and conclude that $x \in \bar{\mathcal{B}}(p_j, \delta)$. □

Remark Constructively, we cannot decide $x \leq 0 \vee x \geq 0$ for an arbitrary real number x. Consequently, we cannot decide whether a point in a Euclidean space belongs to a given set. However, we can compare a real number with a non–trivial interval. In the lemma above, we generalize this result and use the fact that the hypercubes $\bar{\mathcal{H}}(p_i, \delta)$ overlap.

Let $\bar{\mathcal{B}}(0, K) \subset \mathbb{R}^n$ be a closed ball centered at the origin, and $P = \{p_i\}_i$ be its mesh with a step δ. Let h denote the side length of closed hypercubes $\bar{\mathcal{H}}_i(p_i, h)$ such that

$$\|\partial \bar{\mathcal{B}}(0, K) - \partial \cup_i \bar{\mathcal{H}}_i(p_i, h)\| = \frac{\delta}{2}.$$

Here, ∂ denotes the boundary of a set. We call the algebraic number $h_\delta := \min\{\delta, h\}$ the **characteristic step** of the mesh $P = \{p_i\}_i$.

Clearly, the union of the closed hypercubes $\bar{\mathcal{H}}_i(p_i, h_\delta)$ contains the closed ball $\bar{\mathcal{B}}(0, K)$ and all closed hypercubes with adjacent centers have non–trivial intersections. The following shows that an arbitrary point can also be located inside a closed ball and is thus a direct consequence of Lemma 2.1:

Lemma 2.2 *Let $P = \{p_i\}_i$ be a regular mesh on a closed ball $\bar{\mathcal{B}}(0, K) \subset \mathbb{R}^n$ with a step δ. Let h_δ denote its characteristic step. Suppose that x is an arbitrary point in $\bar{\mathcal{B}}(0, K)$. Then, there exists a closed ball $\bar{\mathcal{B}}(p_i, \sqrt{n}h_\delta)$, $p_i \in P$ such that $x \in \bar{\mathcal{B}}(p_i, \sqrt{n}h_\delta)$.*

2.3 Functions and functionals

Let X denote a closed ball or hypercube in $\mathbb{R}^n$. A uniformly continuous (vector–valued) **function** from X to a $\mathbb{R}^m$ is a pair consisting of an operation $x \mapsto f(x)$, $x \in X$ and an operation $\omega\colon\ \mathbb{Q} \to \mathbb{Q}$ called **modulus of (uniform) continuity** such that

$$\forall \varepsilon \in \mathbb{Q}. \forall x, y \in X. \|x - y\| \le \omega(\varepsilon) \implies \|f(x) - f(y)\| \le \varepsilon.$$

Notice that there is no restriction in considering ω and ε as rational numbers since they are dense in the reals. If there exists a rational number L such that $\forall x, y \in X. \|f(x) - f(y)\| \le L\|x - y\|$, then the function f is **Lipschitz** continuous. We denote the normed space of all uniformly continuous vector–valued functions from X to $\mathbb{R}^m$ by $\mathcal{C}(X, \mathbb{R}^m)$. The corresponding norm is defined as $\|f\| \triangleq \sup_{x \in X}\|f(x)\|$, which exists constructively since X is a compact set and every f is uniformly continuous. In the current work, we focus on subsets of $\mathcal{C}(X, \mathbb{R}^m)$, $m = 2$ of all Lipschitz–continuous functions that have a common bound and a common Lipschitz constant. In justifying this assumption, we may claim that in a concrete application, the physical signals are limited in magnitude and rate of change. For example, consider the problem of optimal robot navigation on a plane. The vector of the robot's speed can be considered as a function of its position. The maximal magnitude of this vector is the maximal speed that the robot can achieve. The rate of change of the vector may be limited by the physical capabilities of the steering mechanisms. In order to evaluate separate control strategies, we require a performance mark. To this end, we use the notion of a **uniformly continuous functional** on some function space $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^m)$, which is a pair consisting of an operation $f \mapsto F[f] \in \mathbb{R}, f \in \mathcal{F}$ and a modulus of continuity α satisfying

$$\forall \varepsilon \in \mathbb{Q}. \forall f, g \in \mathcal{F}. \|f - g\| \le \alpha(\varepsilon) \implies |F[f] - F[g]| \le \varepsilon.$$

3 Proof of the main theorem

In this section, we prove Theorem 1.2. Let us recall its statement:

Let $\mathcal{F}$ be the space of all uniformly Lipschitz and uniformly bounded functions from a closed hypercube $X \subset \mathbb{R}^n$ to the plane $\mathbb{R}^2$, and let J be a uniformly continuous functional from $\mathcal{F}$ to $\mathbb{R}$. Then, for any $k \in \mathbb{N}$, there exists an $f^* \in \mathcal{F}$ such that $J[f^*] - \frac{1}{k} \leq \inf J$.

Proof The proof is organized into two parts. In the first, and the crucial, part, it is shown that the function space $\mathcal{F}$ is totally bounded. It is done by choosing a regular mesh on the target plane. It is then shown that any function from the said function space can be approximated by a piece-wise function which does not violate the Lipschitz constant. To this end, the Brehm's extension theorem is used [4]. Provided that the points in question possess solely rational coordinates, its proof is within the Bishop's constructive mathematics. This is verified in the appendix. Finally, to achieve the required approximation precision, the step of the regular mesh needs to be chosen with special care. The second part is simpler and shows existence of the approximate extrema. The details are given below.

Part 1

Let L and K be the uniform Lipschitz constant and uniform bound for $\mathcal{F}$ respectively, and let $X_0 = \{x_i\}_{i=1}^N$ be a finite sequence of unequal points in X. Without loss of generality, we may assume that all the points $(x_1, \ldots, x_N)$ possess solely rational coordinates. We want to show that the subset

$$\Theta := \{(f(x_1), \ldots, f(x_N)) : f \in \mathcal{F}\}$$

of $\mathbb{R}^{2\times N}$ with the metric

$$\|(f(x_1), \ldots, f(x_N)) - (g(x_1), \ldots, g(x_N))\|_\Theta := \sum_{i=1}^{N} \|f(x_i) - g(x_i)\|$$

is *totally bounded* in the following sense: for any $k \in \mathbb{N}$, there exists a finite set $\{\theta_i\}_i$ of unequal elements of Θ such that for any $\theta \in \Theta$, there exists a $\theta_j \in \{\theta_i\}$ such that $\|\theta - \theta_j\|_\Theta \leq \frac{1}{k}$. To this end, let $P = \{p_i\}_i$ be a regular mesh on the closed ball $\bar{\mathcal{B}}(0, K) \subset \mathbb{R}^2$ with a step δ and characteristic step h_δ. Let f be an arbitrary function in the space $\mathcal{F}$. Fix an arbitrary $l \in \mathbb{N}$. We construct an approximating function $\psi\colon X \to \mathbb{R}^2$ such that $\forall x, y \in X. \|\psi(x) - \psi(y)\| \leq L\|x - y\|$ and $\forall x_i \in X_0. \|f(x_i) - \psi(x_i)\| \leq \frac{1}{lN}$. First, the image of ψ on X_0 is constructed inductively. By Lemma 2.2, there exists a closed ball $\bar{\mathcal{B}}(p_{j_1}, \sqrt{2}h_\delta), p_{j_1} \in P$ that contains $f(x_1)$. We set $\psi(x_1) := p_{j_1}$. Assume that $f(x_2) \in \bar{\mathcal{B}}(p_{j_2}, r_1), p_{j_2} \in P,\ r_1 := \sqrt{2}h_\delta$. We pick a point p'_{j_2} closest to p_{j_2} such that

$$\|p_{j_1} - p'_{j_2}\| \le L\|x_1 - x_2\|$$

which, by setting $\psi(x_1) := p'_{j_2}$, satisfies the Lipschitz condition for ψ. At this step, we may take a closed ball $\bar{\mathcal{B}}(p'_{j_2}, r_2)$ that contains $\bar{\mathcal{B}}(p_{j_2}, r_1)$, and in turn $f(x_2)$, with a radius r_2 that is larger than r_1 by a value that depends only on the choice of the initial mesh step δ. Since the total number of steps is fixed by N, setting δ sufficiently small ensures that all $\psi(X_0)$ are within $\frac{1}{lN}$ from the respective $f(X_0)$ while preserving the Lipschitz constant. Now, we need to extend ψ to the whole X. First, we may assume that $L = 1$ without loss of generality since, otherwise, we can scale the X accordingly. If $n > 2$, we project the points of X onto $\mathbb{R}^2$. Denote this projection by π_2. Notice that π_2 has a Lipschitz constant equal to one and, clearly, it projects points with rational coordinates onto points with rational coordinates. Then, we apply Theorem 4.1 from Section 4 to extend $\pi_2 \circ \psi$ onto the entire X. Precomposing the resulting function with the projection onto the ball $\bar{\mathcal{B}}(0, K)$ ensures that it belongs to the space $\mathcal{F}$. If $n \le 2$, the extension theorem applies directly and no projection onto the two–dimensional subspace is required. Let us denote the resulting function by ψ.

So far, we showed how to construct an arbitrarily close approximation to $f \in \mathcal{F}$. Since each such an approximating function is uniquely defined by its values at finitely many points X_0, and since the distances between the approximating function values at each two points of X_0 have fixed bounds, there are finitely many such functions. Further, since f was arbitrary, it follows that Θ is totally bounded. By the constructive Arzela–Ascoli's lemma (Bishop and Bridges [3, page 100]), the function space $\mathcal{F}$ is totally bounded as well. Next, we find an approximate extremum of J.

Part 2

Since $\mathcal{F}$ is totally bounded and J is uniformly continuous, $\inf J$ exists. Let α be the continuity modulus of J and $\mathcal{F}_0 = \{f_1, \dots, f_N\}$ be an $\alpha(\frac{1}{8k})$–approximation to $\mathcal{F}$. Consider all finitely many $\{J[f_i](8k)\}, i = 1, \dots, N$. Let $J[f_j](8k)$ be the smallest one, and such that j is the smallest index if there are more than one such indices. Observe that $|J[f_j] - J[f_j](8k)| \le \frac{1}{4k}$ and $\forall f \in \mathcal{F}. \|f_j - f\| \le \alpha(\frac{1}{8k}) \implies |J(f) - J[f_j]| \le \frac{1}{4k}$ whence $J[f_j](8k) - \frac{1}{2k} \le J[f](8k)$. Therefore, $J[f_j](8k) - \frac{1}{2k} \le J[f]$ and consequently $J[f_j] - \frac{1}{k} \le J[f]$. The same holds trivially if $\|f_j - f\| > \alpha(\frac{1}{8k})$. Since f is arbitrary, $J[f_j] - \frac{1}{k}$ is a lower bound of J and so, in particular, $\inf J \ge J[f_j] - \frac{1}{k}$. $\square$

4 Conclusion

In the present work, we addressed the extremum value theorem for spaces of two–dimensional vector–valued functions on Euclidean spaces from the standpoint of constructive mathematics. By showing that spaces of uniformly Lipschitz and uniformly bounded functions from a closed hypercube to a plane are totally bounded, an approximate extremum of the respective uniformly continuous functional can be constructed. The theorem proven may be used for purposes of formal verification in certain applications such as optimal control.

Appendix

We call a point in a Euclidean space $\mathbb{R}^n$ **algebraic** if its coordinates are solely algebraic numbers.

A (closed) **polyhedron** is an inhabited set of points of the Euclidean space satisfying linear inequalities $Ax \leq b, A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n \times 1}$. If the entries of A and b are solely algebraic numbers, then the polyhedron is called *algebraic*. If a polytope is a union of solely algebraic polyhedrons, then it is algebraic as well. A polyhedron (respectively, polytope) P is **bounded** if there exists a rational number $\bar{x}$ such that $\|x\| \leq \bar{x}$ for any x in P. An n–dimensional **simplex** is a convex hull of $n+1$ affinely independent points. A **triangulation** of a bounded algebraic n–dimensional polyhedron P is a finite set $\{T_i\}_i$ of algebraic simplices whose intersections are at most $(n-1)$–dimensional and such that $P = \cup_i T_i$. For example, a triangulation of a two–dimensional polyhedron is a collection of non–degenerate triangles that may have a common vertex or segment of an edge, but no two–dimensional intersection.

A **motion** in the plane $\mathbb{R}^2$ is a function f that is a composition of a translation, rotation and reflection. Clearly, it is **distance–preserving** in the sense that $\|f(x)-f(y)\| = \|x-y\|$ for any x, y. Translation, rotation and reflection can be described as linear transformations of the form $x \mapsto Tx$ where T is the transformation matrix. A motion can be thus described by a transformation matrix as well. Notice that a motion is always invertible since the corresponding transformation matrix is regular. A motion is called *algebraic* if the corresponding transformation matrix comprises solely of algebraic entries. For example,

$$f(x) = \begin{bmatrix} \sqrt{1-\frac{9}{25}} & \frac{3}{5} \\ -\frac{3}{5} & \sqrt{1-\frac{9}{25}} \end{bmatrix} x$$

is an algebraic motion and describes the clockwise rotation by the angle $\arcsin \frac{3}{5}$. In contrast, $f(x) = x + \pi$ is not an algebraic motion. Any three algebraic points forming a non–degenerate triangle can be moved by an algebraic motion to new algebraic points preserving the respective distances (Petrunin and Yashinski [8]). An **algebraic piecewise motion** f on a bounded algebraic two–dimensional polyhedron P is a pair of a triangulation $\{T_i\}_i$ of the polyhedron and a collection of algebraic motions $\{f_i\}_i$ such that $f|_{T_i} = f_i$. For example, folding of a piece of paper without ripping can be considered as an algebraic piecewise motion if foldings are performed at algebraic points. Notice that each algebraic piecewise motion has a triangulation and a collection of algebraic motions as witnesses. Clearly, an algebraic piecewise motion is **non–expansive** in the sense that $\|f(x) - f(y)\| \leq \|x - y\|$ for any x, y. The notion of an algebraic piecewise motion can be directly generalized to algebraic polytopes.

Constructively, we do not have the full power of set operations in a Euclidean space. For example, if A and B are arbitrary sets in $\mathbb{R}^n$, we cannot decide whether $A \cap B = \emptyset$ or $A \cap B$ is inhabited. In the current work, we are not concerned with arbitrary set operations and limit ourselves to the class of semi-algebraic sets of the form $\left\{x : \bigvee_{i=1}^{N} \bigwedge_{j=1}^{M_i} \mathcal{E}_{ij}\right\}$ with $\mathcal{E}_{ij}$ being a formula of the type $A_{ij}x \bullet b_{ij}$ or $\|f_{ij}(x)\| \bullet \|g_{ij}(x)\|$ where "$\bullet$" denotes "$<$","$\leq$" or "$=$" and $f_{ij}(x)$ and $g_{ij}(x)$ are algebraic piecewise motions on algebraic polytopes. Let us denote this class by $\mathcal{AS}$. For example, an algebraic polytope itself belongs to $\mathcal{AS}$. Further, the set complement of a set $A \in \mathcal{AS}$, denoted by $\mathbb{R}^n \backslash A$, is, again, a set of the class $\mathcal{AS}$ (it can be done by transforming the sign "$<$","$\leq$" or "$=$" in the respective formula). Notice that if $f_{ij}(x)$ and $g_{ij}(x)$ are algebraic motions, each $\|f_{ij}(x)\| \leq \|g_{ij}(x)\|$ is equivalent to $f_{ij}^2(x) \leq g_{ij}^2(x) \wedge f_{ij}(x) \geq 0 \wedge g_{ij}(x) \geq 0$ which is a collection of algebraic inequalities. The same applies if $f_{ij}(x)$ and $g_{ij}(x)$ are algebraic piecewise motions by considering the inequalities on the simplices where $f_{ij}(x)$ and $g_{ij}(x)$ are both algebraic motions. Working solely with sets of the class $\mathcal{AS}$ allows us to perform the ordinary set operations via solving the respective systems of algebraic equations and inequalities. A similar strategy of working with semi-algebraic sets was already used in formal verification of control systems by Platzer [9] based on the quantifier elimination on real closed fields Tarski [11]. Working in $\mathcal{AS}$ will be used in Section 4 to verify that certain vector–valued functions on finite subsets of $\mathbb{R}^2$ admit non–expansive extensions. It is shown that the Brehm's extension theorem is valid in the Bishop's constructive mathematics provided that the points in question are rational.

Extension theorem

For the proof of Theorem 1.2, we require the following result:

Theorem 4.1 *Let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ be finite subsets of points in $\mathbb{R}^2$ with rational coordinates such that $\forall i, j. \|b_i - b_j\| \leq \|a_i - a_j\|$. Let A be the convex hull of $\{a_i\}_{i=1}^n$ and X be a closed hypercube containing A. Then, there exists a non–expansive function $f : X \to \mathbb{R}^2$ such that $\forall i. f(a_i) = b_i$.*

This theorem was first addressed by Kirszbraun [7] and Valentine [12], and then revisited by Brehm [4]. A similar proof can be found in Akopyan and Tarasov [1] and Petrunin and Yashinski [8, page 21]. In our restricted case, we consider only points with rational coordinates. In the following, we verify that the proof is constructive using the notions introduced in Section 2.2.

Proof We first construct an algebraic distance–preserving function h on A with the required properties by induction on the number of the points. If $n = 1$, we may take $h(x) := x + (b_1 - a_1)$ which is clearly an algebraic motion on the entire space. Suppose that an algebraic piecewise motion $g : A \to \mathbb{R}^2$, such that $\forall i = 1, \dots, n-1. g(a_i) = b_i$, was constructed. Define a set $\Omega := \{x : x \in A \wedge \|a_n - x\| < \|b_n - g(x)\|\}$. Since g is algebraic, we can decide whether $b_n = g(a_n)$ or $b_n \neq g(a_n)$. In the former case, we take h to be g. In the latter, Ω is inhabited since $a_n \in \Omega$. We show that if x belongs to Ω, then so does the line segment between a_n and x. Take a point y in this line segment. Then

$$\|a_n - y\| + \|y - x\| = \|a_n - x\|.$$

Since $x \in \Omega$,

$$\|a_n - x\| \leq \|b_n - g(x)\|.$$

The function g is an algebraic piecewise motion which implies

$$\|g(x) - g(y)\| \leq \|x - y\|.$$

Therefore,

$$\begin{aligned} \|a_n - y\| &= \|a_n - x\| - \|y - x\| \\ &< \|b_n - g(x)\| - \|g(x) - g(y)\| \\ &\leq \|b_n - g(y)\| \end{aligned} \tag{2}$$

where the last line follows from the triangle inequality $\|g(x) - g(y)\| \leq \|b_n - g(y)\| + \|b_n - g(y)\|$. Since $\|a_n - y\| < \|b_n - g(y)\|$, it follows that $y \in \Omega$. We now inspect how the boundary $\partial_A \Omega := \partial \Omega \cap A$ is constructed. Let $\{T_i\}_i$ be the triangulation of A such that g on each triangle is a motion g_i. Let $c_n := g_i^{-1}(a_n)$. Notice that c_n is an algebraic point. Since g_i is a motion and $g|_{T_i} = g_i$, for any $x \in T_i$, we have

$$\|c_n - x\| = \|b_n - g(x)\|. \tag{3}$$

Since Ω and T_i belong to $\mathcal{AS}$, we can decide whether the intersection $\Omega \cap T_i$ is inhabited. Suppose it is inhabited. Consider the line

$$l_i := \{x : \|x - a_n\| = \|x - c_n\|\} .$$

We have

$$\partial_A \Omega \cap T_i = \{x : \|a_n - x\| = \|b_n - g(x)\| \wedge x \in T_i \wedge x \in A\} \tag{4}$$

and

$$l_i \cap T_i \cap A = \{x : \|x - a_n\| = \|x - c_n\| \wedge x \in T_i \wedge x \in A\} . \tag{5}$$

Matching (4) with (5) using (3), we see that $\partial_A \Omega \cap T_i$ is a line segment. Since $\{T_i\}_i$ is a finite set, $\partial_A \Omega$ is a finite collection of line segments. Consider a line segment ω_i of $\partial_A \Omega$. Let τ_i be the triangle formed by a_n and ω_i. Let h_i be an algebraic motion that maps a_n to b_n and the endpoints of ω_i to their respective positions under g_i. For $x \in \omega_i$, we have $g(x) = g_i(x)$ and so $g(x) = h_i(x)$. Let $h|_{\tau_i} := h_i$ and $h|_{A \setminus \Omega} := g$. Further, since $\partial\Omega, \partial_A \Omega \in \mathcal{AS}$, we can decide whether $\Delta := \partial\Omega \cap \partial_A \Omega$ is inhabited. If this is the case (for otherwise, we are done), consider the algebraic polytopes $D_k, k = 1, \dots, m$ formed by the endpoints of Δ that lie on $\partial\Omega \cap A$ and the line segments from these endpoints to a_n. Let λ_1 and λ_2 denote the said endpoints for some algebraic polytope D_k. Since h coincides with g on the line segments $[a_n, \lambda_1]$ and $[a_n, \lambda_2]$, and, moreover, it acts as algebraic motions on these line segments, and since g is non–expansive, it follows that:

$$\begin{aligned} \|\lambda_1 - a_n\| =& \|g(\lambda_1) - b_n\| \\ \|\lambda_2 - a_n\| =& \|g(\lambda_2) - b_n\| \\ \|g(\lambda_1) - g(\lambda_2)\| \leq& \|\lambda_1 - \lambda_2\| \end{aligned}$$

We can construct the required function on D_k as follows. Translate and rotate D_k so that $[a_n, \lambda_1]$ coincides with $[b_n, g(\lambda_1)]$. This can be done since the initial and the new vertices of D_k are algebraic. So far, the line segment $[a_n, \lambda_2]$ "turned" around b_n closer to $[a_n, \lambda_1]$. Draw a line segment from $g(\lambda_1)$ to $g(\lambda_2)$ which is the chord of the circle on which the point λ_2 slid to the new position. Take the middle point of the chord and fold D_k around the ray going from a_n to this middle point so that λ_2 matches with $g(\lambda_2)$. The resulting function is thus constructed by translating and rotating the whole D_k and then reflecting the fragment–to–fold around the said ray which constitutes a piecewise motion. This motion is clearly algebraic since all the points involved are algebraic. So far, we constructed an algebraic piecewise motion $h : A \to \mathbb{R}^2$ such that $\forall i. h(a_i) = b_i$. To construct a required function f on the whole X, observe that the projection onto an algebraic polyhedron is non–expansive. In showing this, we use the fact that an algebraic polyhedron is convex: suppose that x and y are points and r and s are their respective projections onto A. Since A is convex, $\forall t \in [0,1].\ tr + (1-t)s \in A$ whence $\|(1-t)r + ts - x\| \geq \|s - x\|$. Differentiating the latter with respect to t yields

$$0 \leq \frac{d}{dt}\|(1-t)r + ts - x\|^2\bigg|_{t=0} = 2\langle s - r, r - x\rangle$$

where $\langle \bullet, \bullet \rangle$ denotes the Euclidean inner product. Therefore, $2\langle s - r, r - x\rangle \geq 0$. The same holds for $\langle r - s, s - y\rangle$. The function $\gamma(t) = \|(1-t)r + tx - ((1-t)s + ty)\|^2$ has the derivative with respect to t at zero being equal to $2\langle r - s, x - r - y + s\rangle$ which is non–negative due to the above two inequalities. It follows that γ is an increasing function and, in particular, $\gamma(1) \geq \gamma(0)$ whence $\|x - y\| \geq \|r - s\|$. Therefore, we complete the construction of f by projecting the points of X onto A. □

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