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A uniform stability principle for dual lattices

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Abstract: We prove a highly uniform *stability* or *almost-near* theorem for dual lattices of lattices $L \subseteq \mathbb{R}^n$. More precisely, we show that, for a vector x from the linear span of a lattice $L \subseteq \mathbb{R}^n$, subject to $\lambda_1(L) \geq \lambda > 0$, to be ε-close to some vector from the dual lattice L' of L, it is enough that the inner products $u\,x$ are δ-close (with $\delta < 1/3$) to some integers for all vectors $u \in L$ satisfying $\|u\| \leq r$, where $r > 0$ depends on n, λ, δ and ε, only. This generalizes an analogous result proved for integral lattices in Mačaj and Zlatoš [17]. The proof is nonconstructive, using the ultraproduct construction and a small amount of nonstandard analysis.

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Informally, a property of objects of a certain kind can be said to be "stable" if objects "almost satisfying" this property are already "close" to objects having the property. For that reason results establishing such a stability are frequently referred to as "almost-near" principles or theorems. By making precise the vague notions "almost satisfying" and "close" various rigorous notions of stability can be obtained. The study of stability of functional equations originates from a question about the stability of additive functions $\mathbb{R} \to \mathbb{R}$ and, more generally, of homomorphisms $G \to H$ between metrizable topological groups, asked by Ulam, cf Mauldin [16] and Ulam [20, 21]. Since that time Ulam's type of stability, modified in various ways, was studied for various (systems of) functional equations; see, eg, Rassias [18] and Székelyhidi [19]. A systematic and general approach to this topic in the realm of compact Hausdorff topological spaces, using nonstandard analysis, was developed by Anderson [1]. The study of stability of the homomorphy property with respect to the compact-open topology was commenced by the second of the present authors in [22], [23], and [24]. The survey article by Boualem and Brouzet [4] reflects some recent developments.

In the present paper we will prove the stability result for dual lattices stated in the Abstract and formulated in a more detailed way in Theorem 5.2, as well as some closely related results. Typically, such an "almost-near" result would be formulated in a weaker

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form, stemming from the following intuitive statement: every vector x from the linear span of a lattice $L \subseteq \mathbb{R}^n$, behaving almost like a vector from the dual lattice L' of L in the sense that all its inner products $u\,x = u_1 x_1 + \ldots + u_n x_n$ with vectors u from a "sufficiently big" subset of L are "sufficiently close" to some integer, is already "arbitrarily close" to a vector $y \in L'$. This vague formulation can be made precise as follows:

Let $L \subseteq \mathbb{R}^n$ be a lattice. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $r > 0$ such that for every $x \in \operatorname{span}(L)$, satisfying $|u\,x|_{\mathbb{Z}} \leq \delta$ for all $u \in L$, $\|u\| \leq r$, there is $y \in L'$ such that $\|x - y\| \leq \varepsilon$.

Here $\operatorname{span}(L)$ denotes the linear subspace of $\mathbb{R}^n$ generated by L,

$$|a|_{\mathbb{Z}} = \min_{c \in \mathbb{Z}} |a - c| = \min\bigl(a - \lfloor a \rfloor, \lceil a \rceil - a\bigr)$$

denotes the distance of the real number a from the set of all integers $\mathbb{Z}$ (with $\lfloor a \rfloor$, $\lceil a \rceil$ being the lower and upper integer part of a, respectively), and $\|x\| = \sqrt{x\,x}$ is the Euclidian norm induced by the usual inner (scalar) product $x\,y$ on $\mathbb{R}^n$.

Such a formulation, however, naturally raises the question how the parameters δ and r depend on the parameters n and ε and some properties of the lattice L. Our Theorem 5.2 is a stronger and more uniform result in the sense that it partly answers this question: one can pick any $\delta \in (0, 1/3)$, then r can be chosen depending on n, ε, δ and, additionally, the Minkowski first successive minimum $\lambda_1(L)$. On the other hand, as the proof of this result uses the ultraproduct construction, it only establishes existence of such r without any estimate of its size.

Theorem 5.2 generalizes an analogous result proved in Mačaj and Zlatoš [17] for integral lattices, replacing the condition $L \subseteq \mathbb{Z}^n$ by introducing an additional parameter $\lambda > 0$ and requiring $\lambda_1(L) \geq \lambda$. The result in [17] was obtained as a byproduct of a stability result for characters of countable Abelian groups the proof of which used Pontryagin-van Kampen duality between discrete and compact groups and the ultraproduct construction. Our present result is based on an intuitively appealing almost-near result (Theorem 4.4) formulated in terms of nonstandard analysis which is linked to its standard counterpart (Theorem 5.2) via the ultraproduct construction applied to a sequence of lattices. As a consequence, Pontryagin-van Kampen duality is eliminated from the proof. Additionally, the passage from stability of characters to stability of dual lattices in [17] naturally led to a formulation in terms of the pair of mutually dual norms $\|x\|_1 = |x_1| + \ldots + |x_n|$ and $\|x\|_\infty = \max\bigl(|x_1|, \ldots, |x_n|\bigr)$. In our present work, starting right away from lattices, the (equivalent) formulation in terms of the (selfdual) Euclidian norm $\|x\| = \|x\|_2$ seems more natural.

1 Lattices and dual lattices

We assume some basic knowledge of lattices or, more generally, of "geometry of numbers". The readers can consult, eg, Cassels [5], Gruber and Lekkerkerker [8] or Lagarias [12]; however, for convenience we list here the definitions of most notions we use and some facts we build on.

A subgroup L of the additive group $\mathbb{R}^n$, where $n \geq 1$, is called a *lattice* if it is discrete, ie there is $\lambda > 0$ such that $\|x - y\| \geq \lambda$ for any distinct vectors $x, y \in L$. $\mathbb{R}^n$ is alternatively viewed as a vector space or an affine space and its elements as vectors or points, respectively. The dimension of the linear space $\operatorname{span}(X)$ generated by a set $X \subseteq \mathbb{R}^n$ is called the *rank* of X ie $\operatorname{rank}(X) = \dim \operatorname{span}(X)$. A *full rank lattice* is a lattice of rank equal the dimension of the ambient space $\mathbb{R}^n$. A *body* is a nonempty bounded connected set $C \subseteq \mathbb{R}^n$ which equals the closure of its interior. A body C is called *centrally symmetric* if $-x \in C$ for any $x \in C$; it is called *convex* if $ax + (1 - a)y \in C$ for any $x, y \in C$ and $a \in [0, 1]$. An example of a centrally symmetric convex body is the Euclidian unit ball $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. The *Minkowski successive minima* of L (with respect to the unit ball B) are defined by

$$\lambda_k(L) = \inf\{\lambda \in \mathbb{R} : \lambda > 0,\ \operatorname{rank}(L \cap \lambda B) \geq k\}$$

for $1 \leq k \leq \operatorname{rank}(L)$. In particular, $\lambda_1(L) = \inf\{\|x\| : 0 \neq x \in L\}$. The *covering radius* of L is defined by

$$\mu(L) = \inf\{r \in \mathbb{R} : r > 0,\ \operatorname{span}(L) \subseteq L + rB\}.$$

In all these cases the infima are in fact minima.

A *basis* of a lattice $L \subseteq \mathbb{R}^n$ is an ordered m-tuple $\beta = (v_1, \ldots, v_m)$ of linearly independent vectors from L which generate L as a group, ie

$$L = \operatorname{grp}(v_1, \ldots, v_m) = \{c_1 v_1 + \ldots + c_m v_m : c_1, \ldots, c_m \in \mathbb{Z}\}.$$

Obviously, in such a case $\operatorname{rank}(L) = m$. In the proof of the fact that every lattice has a basis the following elementary lemma, to which we will refer shortly, plays a key role.

Lemma 1.1 *Let $L \subseteq \mathbb{R}^n$ be a lattice of rank m and $(v_1, \ldots, v_k)$, with $k < m$, be an ordered k-tuple of linearly independent vectors from L which can be extended to a basis of L. Denote $V = \operatorname{span}(v_1, \ldots, v_k)$ and assume that the vector $v_{k+1} \in L \smallsetminus V$ has a minimal (Euclidian) distance to the linear subspace V from among all the vectors in $L \smallsetminus V$. Then the $(k+1)$-tuple $(v_1, \ldots, v_k, v_{k+1})$ either is already a basis of L (if $k + 1 = m$) or it can be extended to a basis of L (if $k + 1 < m$).*

We will use the following consequence of the fact that every lattice has a basis.

Lemma 1.2 *Let $L \subseteq \mathbb{R}^n$ be a lattice. Then a k-tuple of vectors $v_1, \dots, v_k \in L$ is linearly independent if and only if, for any* integers *$c_1, \dots, c_k \in \mathbb{Z}$, the equality $c_1 v_1 + \dots + c_k v_k = 0$ implies $c_1 = \dots = c_k = 0$.*

A basis $(v_1, \dots, v_m)$ of a lattice L is *Minkowski reduced* if, for each $k \le m$, v_k is the shortest vector from L such that the k-tuple $(v_1, \dots, v_k)$ can be extended to a basis of L. It is known that every lattice has a Minkowski reduced basis.

For any subset $S \subseteq \mathbb{R}^n$ we denote by

$$\mathrm{Ann}_{\mathbb{Z}}(S) = \{x \in \mathbb{R}^n : \forall\, u \in S \colon u\,x \in \mathbb{Z}\}$$

the *integral annihilator* of S. Obviously, $\mathrm{Ann}_{\mathbb{Z}}(S)$ is a subgroup of $\mathbb{R}^n$ for every $S \subseteq \mathbb{R}^n$; however, even for a lattice $L \subseteq \mathbb{R}^n$, the integral annihilator $\mathrm{Ann}_{\mathbb{Z}}(L)$ need not be a lattice unless $\mathrm{rank}(L) = n$. The *dual lattice* of L (also called the *polar* or *reciprocal lattice*) is defined as the intersection

$$L' = \mathrm{Ann}_{\mathbb{Z}}(L) \cap \mathrm{span}(L)\,.$$

Then L' is a lattice in $\mathbb{R}^n$ of the same rank as L and there is an obvious duality relation $L'' = L$. The Minkowski successive minima of the original lattice L and its dual lattice L' are related through a bound due to Banaszczyk [2]. Similarly, the covering radius of the dual lattice L' can be estimated in terms of the first Minkowski minimum of L; see Lagarias, Lenstra and Schnorr [13] or Lagarias [12]. Actually, in the quoted papers these results were stated and proved for full rank lattices, ie only in case $m = n$. However, introducing an orthonormal basis in the linear subspace $\mathrm{span}(L)$ and replacing any vector $x \in \mathrm{span}(L)$ by its coordinates with respect to it, they can be readily generalized as follows.

Lemma 1.3 *Let $L \subseteq \mathbb{R}^n$ be a lattice of rank m. Then*

$$\lambda_k(L)\, \lambda_{m-k+1}\big(L'\big) \le m \qquad \text{for each } k \le m,$$

and

$$\lambda_1(L)\, \mu\big(L'\big) \le \frac{1}{2}\, m^{3/2}.$$

2 Ultraproducts of lattices

In order to keep our presentation self-contained, we give a brief account of the ultraproduct construction and some notions of nonstandard analysis here. Nonetheless, the readers are strongly advised to consult some more detailed exposition such as those in Chang-Keisler [6], Davis [7] and Henson [10].

A nonempty system D of subsets of a set I is a called a *filter* on I if $\emptyset \notin D$, D is closed with respect to intersections, and, for any $X \in D$, $Y \subseteq I$, the inclusion $X \subseteq Y$ implies $Y \in D$. A filter D on I is called an *ultrafilter* if for any $X \subseteq I$ either $X \in D$ or $I \smallsetminus X \in D$. Ultrafilters of the form $D = \{X \subseteq I : j \in X\}$, where $j \in I$, are called *principal*. As a consequence of the *axiom of choice*, every filter on I is contained in some ultrafilter; in particular, nonprincipal ultrafilters exist on every infinite set I.

Given a set I and a family of first order structures $(A_i)_{i\in I}$ of some first order language Λ, we can form their direct product $\prod_{i\in I} A_i$ with basic operations and relations defined componentwise. If, additionally, D is a filter on I, then

$$\alpha \equiv_D \beta \iff \{i \in I : \ \alpha(i) = \beta(i)\} \in D$$

defines an equivalence relation on $\prod A_i$. Denoting by α/D the coset of a function $\alpha \in \prod A_i$ with respect to $\equiv_D$, the quotient

$$B = \prod A_i \Big/ D = \prod A_i \Big/ {\equiv_D}\,,$$

naturally becomes a Λ-structure once we define

$$f^B(\alpha_1/D, \dots, \alpha_p/D) = \beta/D\,,$$

where $\beta(i) = f^{A_i}(\alpha_1(i), \dots, \alpha_p(i))$, for any p-ary functional symbol f, and

$$(\alpha_1/D, \dots, \alpha_p/D) \in R^B \iff \{i \in I : (\alpha_1(i), \dots, \alpha_p(i)) \in R^{A_i}\} \in D$$

for any p-ary relational symbol R. Then B is called the *filtered* or *reduced product* of the family (A_i) with respect to the filter D. If $A_i = A$ is the same structure for each $i \in I$, then the reduced product

$$A^I/D = \prod A_i \Big/ D$$

is called the *filtered* or *reduced power* of the Λ-structure A. If D is an ultrafilter, then we speak of *ultraproducts* and *ultrapowers*.

The following property of ultraproducts is of fundamental importance.

Lemma 2.1 (Łos Theorem) *Let $(A_i)_{i\in I}$ be a family of structures of some first order language Λ, D be an ultrafilter on the index set I, $\Phi(x_1, \dots, x_p)$ be a Λ-formula and $\alpha_1, \dots, \alpha_p \in \prod A_i$. Then the statement $\Phi(\alpha_1/D, \dots, \alpha_p/D)$ holds in the ultraproduct $\prod A_i/D$ if and only if*

$$\{i \in I : \ \Phi(\alpha_1(i), \dots, \alpha_p(i)) \text{ holds in } A_i\} \in D\,.$$

As a consequence, the canonical embedding of any Λ-structure A into its ultrapower ${}^*A = A^I/D$ is *elementary*. More precisely, identifying every element $a \in A$ with the coset $\bar{a}/D$ of the constant function $\bar{a}(i) = a$, we have

$$\Phi(a_1, \dots, a_p) \text{ holds in } A \iff \Phi(a_1, \dots, a_p) \text{ holds in } {}^*A$$

for every Λ-formula $\Phi(x_1, \dots, x_n)$ and any $a_1, \dots, a_p \in A$. This equivalence will be referred to as the *transfer principle*.

The above accounts almost directly apply to many-sorted structures, like modules over rings or vector spaces over fields as well (see Henson [10]). In particular, if $(V_i)_{i\in I}$ is a family of vector spaces over a field F, then the ultraproduct $\prod V_i/D$ becomes a vector space over the ultrapower ${}^*F = F^I/D$, which is a field elementarily extending F. Similarly, if $(G_i)_{i\in I}$ is a family of Abelian groups, viewed as modules over the ring of integers $\mathbb{Z}$, then the ultraproduct $\prod G_i/D$ becomes not only an Abelian group, but also a module over the ring of *hyperintegers* ${}^*\mathbb{Z} = \mathbb{Z}^I/D$ elementarily extending the ring $\mathbb{Z}$. And, what is of crucial importance, the Łos Theorem is still true for formulas in the corresponding two-sorted language.

From now on $I = \{1, 2, 3, \dots\}$ denotes the set of all positive integers and D is some fixed nonprincipal ultrafilter on I. We form the ordered field of *hyperreal numbers* as the ultrapower ${}^*\mathbb{R} = \mathbb{R}^I/D$ of the ordered field $\mathbb{R}$. Then

$$\mathbb{F}^*\mathbb{R} = \{x \in {}^*\mathbb{R} : \exists\, r \in \mathbb{R},\ r > 0\colon\ |x| < r\}$$

and

$$\mathbb{I}^*\mathbb{R} = \{x \in {}^*\mathbb{R} : \forall\, r \in \mathbb{R},\ r > 0\colon\ |x| < r\}.$$

denote the sets of all *finite* hyperreals and of all *infinitesimals*, respectively. It can be easily verified that $\mathbb{F}^*\mathbb{R}$ is a subring of ${}^*\mathbb{R}$ and $\mathbb{I}^*\mathbb{R}$ is an ideal in $\mathbb{F}^*\mathbb{R}$. Hyperreal numbers not belonging to $\mathbb{F}^*\mathbb{R}$ are called *infinite*. For $x \in {}^*\mathbb{R}$ we sometimes write $|x| < \infty$ instead of $x \in \mathbb{F}^*\mathbb{R}$, and $x \sim \infty$ instead of $x \notin \mathbb{F}^*\mathbb{R}$. Two hyperreals x, y are said to be *infinitesimally close*, in notation $x \approx y$, if $x - y \in \mathbb{I}^*\mathbb{R}$. Moreover, for each $x \in \mathbb{F}^*\mathbb{R}$, there is a unique real number ${}^\circ x \in \mathbb{R}$, called the *standard part* of x, such that $x \approx {}^\circ x$. As a consequence, $\mathbb{F}^*\mathbb{R}/\mathbb{I}^*\mathbb{R} \cong \mathbb{R}$ as ordered fields.

A hyperreal number $x = \alpha/D$, where $\alpha\colon I \to \mathbb{R}$, is finite if and only if there is a positive $r \in \mathbb{R}$ such that $\{i \in I : |\alpha(i)| < r\} \in D$; this is equivalent to the convergence of the sequence α to ${}^\circ x$ with respect to the filter D. In particular, x is infinitesimal if and only if ${}^\circ x = 0$, ie if and only if the sequence α converges to 0 with respect to D. As D necessarily extends the Frechet filter, $\lim_{i\to\infty} \alpha(i) = a \in \mathbb{R}$ in the usual sense implies ${}^\circ x = a$, ie $x \approx a$.

The standard part map has the following homomorphy properties with respect to the field operations:

$${}^\circ(x + y) = {}^\circ x + {}^\circ y \quad \text{and} \quad {}^\circ(x\,y) = {}^\circ x\,{}^\circ y$$

for any $x, y \in \mathbb{F}^*\mathbb{R}$, and if additionally $x \not\approx 0$, then also ${}^\circ\big(x^{-1}\big) = \big({}^\circ x\big)^{-1}$.

Along with the equivalence relation of infinitesimal nearness $\approx$, we introduce the relation of *Archimedean equivalence* $\sim$ or *order equality* on ${}^*\mathbb{R}$ as follows:

$$x \sim 0 \;\Leftrightarrow\; x = 0\,, \quad \text{and} \quad x \sim y \;\Leftrightarrow\; 0 \not\approx \frac{x}{y} \in \mathbb{F}^*\mathbb{R} \;\text{ for } x, y \neq 0\,.$$

When $x \sim y$ we say that x and y are of the *same (Archimedean) order*. We also say that x is of *smaller order* than y, or that y is of *bigger order* than x, in symbols $x \ll y$, if $y \neq 0$ and $\frac{x}{y} \approx 0$. Obviously, for $x, y \neq 0$, $x \sim y$ is equivalent to: neither $x \ll y$ nor $y \ll x$.

According to the transfer principle, we can identify, for any finite integer $n \geq 1$, the vector space $({}^*\mathbb{R})^n$ over the field ${}^*\mathbb{R}$ and the ultrapower ${}^*(\mathbb{R}^n) = (\mathbb{R}^n)^I\big/D$, so that the notation ${}^*\mathbb{R}^n$ is unambiguous. More generally, for any subset $S \subseteq \mathbb{R}^n$ we identify the ultrapower ${}^*S = S^I\big/D$ with the subset

$$\{(\alpha_1/D, \dots, \alpha_n/D) \in {}^*\mathbb{R}^n : \{i \in I : (\alpha_1(i), \dots, \alpha_n(i)) \in S\} \in D\}$$

of ${}^*\mathbb{R}^n$. The inner product on $\mathbb{R}^n$ extends to the inner product on ${}^*\mathbb{R}^n$, preserving all its first order properties. In order to distinguish the linear spans with respect to the fields $\mathbb{R}$ and ${}^*\mathbb{R}$, respectively, we introduce the *internal linear span* of a set $X \subseteq {}^*\mathbb{R}^n$ which, due to the fact that the ambient vector space ${}^*\mathbb{R}^n$ has finite internal dimension n, can be described as follows:

$${}^*\mathrm{span}(X) = \{a_1x_1 + \ldots + a_nx_n : x_1, \dots, x_n \in X,\; a_1, \dots, a_n \in {}^*\mathbb{R}\}$$

We also distinguish the lattice or subgroup, ie the $\mathbb{Z}$-submodule $\mathrm{grp}(v_1, \dots, v_m)$ of $\mathbb{R}^n$ generated by vectors $v_1, \dots, v_m \in \mathbb{R}^n$, and the internal lattice *internally generated* by vectors $v_1, \dots, v_m \in {}^*\mathbb{R}^n$, ie the ${}^*\mathbb{Z}$-submodule

$${}^*\mathrm{grp}(v_1, \dots, v_m) = \{c_1v_1 + \ldots + c_mv_m : c_1, \dots, c_m \in {}^*\mathbb{Z}\}$$

of ${}^*\mathbb{R}^n$.

Similarly as in ${}^*\mathbb{R}$, vectors from $\mathbb{F}^*\mathbb{R}^n$ are called *finite* and vectors from $\mathbb{I}^*\mathbb{R}^n$ are called *infinitesimal*. Obviously

$$\mathbb{F}^*\mathbb{R}^n = \{x \in {}^*\mathbb{R}^n : \|x\| < \infty\}$$

and

$$\mathbb{I}^*\mathbb{R}^n = \{x \in {}^*\mathbb{R}^n : \|x\| \approx 0\}.$$

Both $\mathbb{F}^*\mathbb{R}^n$ and $\mathbb{I}^*\mathbb{R}^n$ are vector spaces over the field $\mathbb{R}$ and even modules over the ring $\mathbb{F}^*\mathbb{R}$, but not over the field ${}^*\mathbb{R}$. Vectors $x, y \in {}^*\mathbb{R}^n$ are said to be *infinitesimally close*, in notation $x \approx y$, if $x - y \in \mathbb{I}^*\mathbb{R}^n$, ie if $\|x - y\| \approx 0$. The *standard part* of a vector $x = (x_1, \dots, x_n) \in \mathbb{F}^*\mathbb{R}^n$ is the vector ${}^\circ x = \big({}^\circ x_1, \dots, {}^\circ x_n\big)$; obviously, ${}^\circ x$ is the unique vector in $\mathbb{R}^n$ infinitesimally close to x. Then $\mathbb{F}^*\mathbb{R}^n/\mathbb{I}^*\mathbb{R}^n \cong \mathbb{R}^n$ as vector spaces over $\mathbb{R}$.

Though the ultraproduct construction can be applied to any family of lattices $L_i \subseteq \mathbb{R}^{n_i}$, it is sufficient for our purpose to deal with lattices situated in the same ambient vector space $\mathbb{R}^n$ with $n \geq 1$ fixed. Given a sequence $(L_i)_{i\in I}$ of lattices $L_i \subseteq \mathbb{R}^n$ we can form the ultraproduct $\prod L_i/D$ and identify it with the subset

$$L = \{(\alpha_1/D, \ldots, \alpha_n/D) \in {}^*\mathbb{R}^n : \{i \in I : (\alpha_1(i), \ldots, \alpha_n(i)) \in L_i\} \in D\}$$

of the vector space ${}^*\mathbb{R}^n$ over ${}^*\mathbb{R}$. Then L is an internal discrete additive subgroup of ${}^*\mathbb{R}^n$, ie it is a module over the ring of *hyperintegers* ${}^*\mathbb{Z}$ and there is a positive $\lambda \in {}^*\mathbb{R}$ such that $\|x - y\| \geq \lambda$ for any distinct $x, y \in L$; however, it should be noticed that λ may well be infinitesimal. Moreover, as D is an ultrafilter, there is $m \leq n$ and a set $J \in D$ such that $\operatorname{rank}(L_i) = m$ for each $i \in J$. We write $\operatorname{rank}(L) = m$ and refer to L as an *internal lattice* in ${}^*\mathbb{R}^n$ of rank m. Then we can assume, without loss of generality, that $\operatorname{rank}(L_i) = m$ for each $i \in I$. The Minkowski successive minima and covering radius of such an internal lattice L can be defined in two ways which are equivalent by the transfer principle:

$$\lambda_k(L) = \big(\lambda_k(L_i)\big)_{i\in I}/D = \min\{\lambda \in {}^*\mathbb{R} : \lambda > 0,\ \operatorname{rank}(L \cap \lambda\, {}^*B) \geq k\},$$

and $$\mu(L) = \big(\mu(L_i)\big)_{i\in I}/D = \min\{r \in {}^*\mathbb{R} : r > 0,\ {}^*\operatorname{span}(L) \subseteq L + r\, {}^*B\}$$

for $k \leq m$. Then $\lambda_1(L) \leq \ldots \leq \lambda_m(L)$ and $\mu(L)$ are positive *hyperreal* numbers, hence they can be both infinitesimals as well as infinite. Additionally, we put

$$\operatorname{rank}_0(L) = \#\{k : 1 \leq k \leq m,\ \lambda_k(L) \approx 0\}$$

and $$\operatorname{rank}_{\mathrm{f}}(L) = \#\{k : 1 \leq k \leq m,\ \lambda_k(L) < \infty\},$$

where $\#H$ denotes the number of elements of a finite set H. Note that $\operatorname{rank}_0(L) = 0$ if $\lambda_1(L) \not\approx 0$, as well as $\operatorname{rank}_{\mathrm{f}}(L) = 0$ if $\lambda_1(L) \notin \mathbb{F}^*\mathbb{R}$. Obviously, if $\operatorname{rank}_0(L) > 0$, then it is the biggest $k \leq m$ such that $\lambda_k(L) \approx 0$; similarly, if $\operatorname{rank}_{\mathrm{f}}(L) > 0$, then it is the biggest $k \leq m$ such that $\lambda_k(L) < \infty$.

At the same time, we can assume that $\beta_1, \ldots, \beta_m \in \prod L_i$ are functions such that, for each $i \in I$ (or at least for each i from some set $J \in D$), the m-tuple of vectors $\boldsymbol{\beta}(i) = (\beta_1(i), \ldots, \beta_m(i))$ is a Minkowski reduced basis of the lattice L_i. Then, due to Łos Theorem (Lemma 2.1), the m-tuple $\boldsymbol{\beta}/D = (v_1, \ldots, v_m)$, where $v_k = \beta_k/D$ for $k \leq m$, is a Minkowski reduced basis of the internal lattice L, ie the vectors $v_1, \ldots, v_m$ are linearly independent over ${}^*\mathbb{R}$ and generate L as a ${}^*\mathbb{Z}$-module.

Lemma 2.2 *Let $L \subseteq {}^*\mathbb{R}^n$ be an internal lattice of rank m and $\boldsymbol{\beta} = (v_1, \ldots, v_m)$ be a Minkowski reduced basis of L. Then the following hold:*

(a) *If, for some* $k < m$, $\|v_k\| \ll \|v_{k+1}\|$ *and* $V = {}^*\mathrm{span}\{v_1, \dots, v_k\}$, *then* $\|x\| \not\ll \|v_{k+1}\|$ *for every vector* $x \in L \smallsetminus V$.

(b) $\|v_k\| \sim \lambda_k(L)$ *for each* $k \leq m$.

Proof (a) Assume that, under the assumptions of (a), we have $\|x\| \ll \|v_{k+1}\|$ for some $x \in L \smallsetminus V$. We denote the orthogonal projection of a vector $y \in {}^*\mathbb{R}^n$ to V by y_V. Let $z \in L \smallsetminus V$ be a vector such that its the distance $z - z_V$ to V is minimal from among all the vectors $y \in L \smallsetminus V$. Therefore,

$$\|z - z_V\| \leq \|x - x_V\| \leq \|x\| \ .$$

As $z_V \in V$, there are hyperreals $a_1, \dots, a_k \in {}^*\mathbb{R}$ such that $z_V = a_1 v_1 + \ldots + a_k v_k$. Denoting by $c_j = \lfloor a_j \rfloor$ their lower integer parts and $z' = z - c_1 v_1 - \ldots - c_k v_k \in L$, we have $z - z' \in V$, hence $\|z' - z'_V\| = \|z - z_V\|$, so that the vector $z' \in L$ has the same minimality property as z. Then, according to Lemma 1.1 and the transfer principle, the $(k+1)$-tuple $(v_1, \dots, v_k, z')$ can be extended to a basis of L, hence $\|v_{k+1}\| \leq \|z'\|$, as the basis $(v_1, \dots, v_m)$ is Minkowski reduced. At the same time,

$$z'_V = (a_1 - c_1)v_1 + \ldots + (a_k - c_k)v_k \ ,$$

with $|a_j - c_j| < 1$ for each $j \leq k$. From the triangle inequality we get:

$$\begin{aligned} \|z'\| &\leq \|z'_V\| + \|z' - z'_V\| \\ &= \|(a_1 - c_1)v_1 + \ldots + (a_k - c_k)v_k\| + \|z - z_V\| \\ &< \|v_1\| + \ldots + \|v_k\| + \|x\| \end{aligned}$$

Therefore, $\|z'\| \ll \|v_{k+1}\|$, hence $\|z'\| < \|v_{k+1}\|$, which is a contradiction.

(b) Because $\|v_1\| = \lambda_1(L)$, the statement of (b) is true for $k = 1$. Assume, toward a contradiction, that $k < m$ for the biggest index satisfying $\|v_k\| \sim \lambda_k(L)$. Then

$$1 \leq \frac{\|v_k\|}{\lambda_k(L)} < \infty \qquad \text{and} \qquad \frac{\lambda_{k+1}(L)}{\|v_{k+1}\|} \approx 0 \, .$$

Therefore,

$$\frac{\|v_k\|}{\|v_{k+1}\|} \leq \frac{\lambda_{k+1}(L)}{\lambda_k(L)} \cdot \frac{\|v_k\|}{\|v_{k+1}\|} = \frac{\|v_k\|}{\lambda_k(L)} \cdot \frac{\lambda_{k+1}(L)}{\|v_{k+1}\|} \approx 0 \, .$$

Then, according to (a), $\frac{\|x\|}{\|v_{k+1}\|} \not\approx 0$ for every vector $x \in L \smallsetminus {}^*\mathrm{span}(v_1, \dots, v_k)$. In particular, $\frac{\lambda_{k+1}(L)}{\|v_{k+1}\|} \not\approx 0$. □

Remark 1 (b) of Lemma 2.2 follows immediately, by applying the transfer principle, from the following estimates of the lengths of vectors in any Minkowski reduced basis $(v_1, \dots, v_m)$ of a rank m lattice $L \subseteq \mathbb{R}^n$ in terms of its Minkowski successive minima:

$$\lambda_k(L) \leq \|v_k\| \leq 2^k \lambda_k(L)$$

for all $k \leq m$ (see Lagarias [11]; Mahler [14] has even better upper bounds). Then (a) could be proved as an easy consequence of (b). However, it is perhaps worthwhile to notice that, using the internal lattice concept, the purely qualitative estimates (a), (b) follow already from Lemma 1.1 and the existence of Minkowski reduced bases.

The *standard part* ${}^\circ X$ of a set $X \subseteq {}^*\mathbb{R}^n$ consists of the standard parts of all finite vectors from X; alternatively, it can be formed by taking the quotient of the set of finite vectors from X with respect to the equivalence relation of infinitesimal nearness. Identifying the results of both approaches, we have

$$ {}^\circ X = \big(X \cap \mathbb{F}^*\mathbb{R}^n\big)/{\approx} = \{{}^\circ x : x \in X \cap \mathbb{F}^*\mathbb{R}^n\} = \{y \in \mathbb{R}^n : \exists x \in X : y \approx x\}\,. $$

In particular, for an additive subgroup $G \subseteq {}^*\mathbb{R}^n$ we denote by

$$ \mathbb{F}G = G \cap \mathbb{F}^*\mathbb{R}^n \qquad \text{and} \qquad \mathbb{I}G = G \cap \mathbb{I}^*\mathbb{R}^n $$

the additive subgroups of ${}^*\mathbb{R}^n$ formed by the finite and infinitesimal elements in G, respectively. Then its standard part ${}^\circ G$ is an additive subgroup of $\mathbb{R}^n$ which can be identified with the quotient

$$ {}^\circ G = \mathbb{F}G/\mathbb{I}G\,. $$

However, even for an internal lattice $L \subseteq {}^*\mathbb{R}^n$, its standard part ${}^\circ L$ is not necessarily discrete, hence it need not be a lattice in $\mathbb{R}^n$. A more detailed account will follow after a preliminary lemma.

Lemma 2.3 *Let $L \subseteq {}^*\mathbb{R}^n$ be an internal lattice of rank m and $\beta = (v_1, \dots, v_m)$ be a Minkowski reduced basis of L such that all the vectors in β are infinitesimal. Then there exist hyperintegers $c_1, \dots, c_m \in {}^*\mathbb{Z}$ such that all the vectors $c_k v_k$ are finite but not infinitesimal and c_k divides c_{k-1} whenever $2 \leq k \leq m$. For such a choice of $c_1, \dots, c_m$ the internal sublattice $M = {}^*\mathrm{grp}(c_1 v_1, \dots, c_m v_m) \subseteq L$ contains no infinitesimal vector except for 0, in other words $\lambda_1(M) \not\approx 0$.*

Proof Let's start with an arbitrary $c_m \in {}^*\mathbb{Z}$ such that $c_m v_m \in \mathbb{F}L \smallsetminus \mathbb{I}L$ (eg, one can put $c_m = \left\lceil \|v_m\|^{-1} \right\rceil$ guaranteeing that $1 \leq \|c_m v_m\| < 1 + \|v_m\| \approx 1$). We proceed by backward recursion. Assuming that $2 \leq k \leq m$ and c_k is already defined, we put $c_{k-1} = c_k$ if $c_k v_{k-1} \not\approx 0$ (as $\|v_{k-1}\| \leq \|v_k\|$, $c_k v_{k-1} \in \mathbb{F}L$ is satisfied automatically), otherwise we put $c_{k-1} = b c_k$ where $b \in {}^*\mathbb{Z}$ is any hyperinteger such that $b c_k v_{k-1} \in \mathbb{F}L \smallsetminus \mathbb{I}L$ (eg, $b = \left\lceil \|c_k v_{k-1}\|^{-1} \right\rceil$ will work). Obviously, $c_k \in {}^*\mathbb{Z}$ divides $c_{k-1} \in {}^*\mathbb{Z}$ for any $2 \leq k \leq m$.

Assume that $x \approx 0$, where $x = a_1 c_1 v_1 + \ldots + a_m c_m v_m$ for some $a_1, \ldots, a_m \in {}^*\mathbb{Z}$ not all equal to 0. Let $q \le m$ be the biggest index such that $a_q \ne 0$. Then

$$x' = \frac{1}{c_q}\, x = \sum_{k=1}^{q} \frac{a_k c_k}{c_q}\, v_k \ne 0$$

is a vector from the internal lattice L. Moreover, $c_q x' = x \approx 0$, while $c_q v_q \not\approx 0$, hence $\|x'\| \ll \|v_q\|$. Let $p \le q$ be the smallest index such that $\|x'\| \ll \|v_p\|$. Denote $\lambda = \|x'\|$ if $p = 1$, or $\lambda = \max(\|v_{p-1}\|, \|x'\|)$ if $p > 1$. Then the hyperball $\lambda\,{}^*B$ contains p linearly independent vectors $v_1, \ldots, v_{p-1}, x'$ from L, hence $\lambda_p(L) \le \lambda$ and, at the same time, $\lambda \ll \|v_p\|$, contradicting Lemma 2.2 (b). $\square$

Proposition 2.4 *Let $L = \prod L_i \big/ D \subseteq {}^*\mathbb{R}^n$ be an internal lattice of rank m and ${}^\circ L$ be its standard part. Then the following hold true:*

(a) ${}^\circ L$ is a lattice in $\mathbb{R}^n$ if and only if there is a positive $\lambda \in \mathbb{R}$ such that the set $\{i \in I : \lambda_1(L_i) \ge \lambda\}$ belongs to D. This is equivalent to $\lambda_1(L) \not\approx 0$ as well as to $\operatorname{rank}_0(L) = 0$.

(b) ${}^\circ L$ is the direct sum of a linear subspace of $\mathbb{R}^n$ of dimension $\operatorname{rank}_0(L)$ and a lattice in $\mathbb{R}^n$ of rank $\operatorname{rank}_{\mathrm{f}}(L) - \operatorname{rank}_0(L)$.

(c) ${}^\circ L$ is a lattice of rank $q \le m$ if and only if $\operatorname{rank}_0(L) = 0$ and $\operatorname{rank}_{\mathrm{f}}(L) = q$.

Proof (a) The equivalence of any of the first two conditions to the discreteness of the group ${}^\circ L$ is obvious. Similarly, any of the obviously equivalent conditions $\lambda_1(L) \not\approx 0$ and $\operatorname{rank}_0(L) = 0$ implies the discreteness of ${}^\circ L$. Otherwise, there is at least one nonzero infinitesimal vector $v \in L$. Then one can find a hyperinteger $c \in {}^*\mathbb{Z}$ such that cv is finite but not infinitesimal. Obviously, its standard part $w = {}^\circ(cv) \ne 0$ belongs to ${}^\circ L$, so that $\operatorname{span}(w) = \mathbb{R}w$ is a line in $\mathbb{R}^n$. We prove the inclusion $\mathbb{R}w \subseteq {}^\circ L$. Taking any $x = aw \in \mathbb{R}w$, with $a \in \mathbb{R}$, and putting $b = \lfloor ac \rfloor \in {}^*\mathbb{Z}$, we have $b \le ac < b + 1$ which, by the virtue of $v \approx 0$, implies $bv \approx acv$. Hence

$$x = aw \approx acv \approx bv \in \mathbb{F}L$$

and $x = {}^\circ(bv) \in {}^\circ L$. It follows that ${}^\circ L$, containing the line $\mathbb{R}w \subseteq \mathbb{R}^n$, is not discrete.

(b) Let $(v_1, \ldots, v_m)$ be a Minkowski reduced basis of L. Denote $p = \operatorname{rank}_0(L)$ and $q = \operatorname{rank}_{\mathrm{f}}(L)$. According to Lemma 2.2 (b), a vector v_k is infinitesimal if and only if $k \le p$, and it is finite if and only if $k \le q$. For the same reason, if $x \in L \smallsetminus {}^*\mathrm{grp}(v_1, \ldots, v_q)$ then $\|x\| \not\ll v_{q+1}$, hence $x \notin \mathbb{F}L$. Therefore the standard part ${}^\circ L$ of the internal lattice L coincides with the standard part of its internal sublattice ${}^*\mathrm{grp}(v_1, \ldots, v_q)$. Due to Lemma 2.3, there are hyperintegers $c_1, \ldots, c_p \in {}^*\mathbb{Z}$ such that

$c_k v_k \in \mathbb{F}L \setminus \mathbb{I}L$ for any k and c_k divides c_{k-1} for $k \geq 2$. Then the internal sublattice $M = {}^*\mathrm{grp}(c_1 v_1, \ldots, c_p v_p) \subseteq L$ contains no nonzero infinitesimal vector. Let us denote $w_k = {}^\circ(c_k v_k)$ for $k \leq p$, and, additionally, $c_k = 1$, $w_k = {}^\circ v_k = {}^\circ(c_k v_k)$ for $p < k \leq q$. As a consequence, ${}^\circ L$ coincides with the sum of the linear subspace $\mathrm{span}(w_1, \ldots, w_p)$ and the lattice $\mathrm{grp}(w_{p+1}, \ldots, w_q)$.

The proof of (b) will be complete once we establish the following claim.

Claim *The vectors $w_1, \ldots, w_q$ are linearly independent over $\mathbb{R}$.*

Indeed, let $b \in {}^*\mathbb{N}$ be any infinite hypernatural number. Put $v'_k = b^{-1} v_k$, $c'_k = b c_k$ for any $k \leq q$. Then all the vectors $v'_1, \ldots, v'_q$ are infinitesimal and form a Minkowski reduced basis of the lattice $L' = \{b^{-1}x : x \in L\}$. Now, all the vectors $c'_k v'_k = c_k v_k$, where $k \leq q$, are finite but not infinitesimal and c'_k divides c'_{k-1} for $k \geq 2$. From Lemma 2.3 we infer that the internal lattice

$$N = {}^*\mathrm{grp}(c_1 v_1, \ldots, c_q v_q) = {}^*\mathrm{grp}\big(c'_1 v'_1, \ldots, c'_q v'_q\big)$$

satisfies $\lambda_1(N) \not\approx 0$. Then, by (a), its standard part ${}^\circ N$ is a lattice in $\mathbb{R}^n$. According to Lemma 1.2, it suffices to show that $a_1 w_1 + \ldots + a_q w_q = 0$ implies $a_1 = \ldots = a_q = 0$ for any *integers* $a_1, \ldots, a_q \in \mathbb{Z}$. Since the first equality is equivalent to $a_1 c_1 v_1 + \cdots + a_q c_q v_q \approx 0$ and the left hand vector belongs to N, which contains no infinitesimal vector except for 0, we have $a_1 c_1 v_1 + \cdots + a_q c_q v_q = 0$, and the desired conclusion follows from the linear independence of the vectors $c_1 v_1, \ldots, c_q v_q$ over ${}^*\mathbb{R}$.

(c) follows directly from (a) and (b). □

The following is a direct consequence of Proposition 2.4(b).

Corollary 2.5 *Let L be an internal lattice in ${}^*\mathbb{R}^n$. Then its standard part ${}^\circ L$ is a closed subgroup of the additive group $\mathbb{R}^n$.*

3 An "almost-near" result for systems of linear equations

We denote by $F^{m\times n}$ the vector space of all $m \times n$ matrices over a field F. Unless otherwise said, the vector space F^n consists of column vectors. The transpose of a matrix A is denoted by $A^\top$. A matrix $A \in {}^*\mathbb{R}^{m\times n}$ is called *finite*, in symbols $A \in \mathbb{F}{}^*\mathbb{R}^{m\times n}$, if all its entries a_{ij} are finite. Then the matrix ${}^\circ A = \big({}^\circ a_{ij}\big) \in \mathbb{R}^{m\times n}$ is called the *standard* *part* of A. The preservation of addition and multiplication by the standard part map on $\mathbb{F}^*\mathbb{R}$ extends to finite matrices, ie

$$^\circ(A+B) = {}^\circ\!A + {}^\circ\!B \quad \text{and} \quad ^\circ(A\,C) = {}^\circ\!A\,{}^\circ C$$

for any $A, B \in \mathbb{F}^*\mathbb{R}^{m\times n}$, $C \in \mathbb{F}^*\mathbb{R}^{n\times p}$.

The following "almost-near" result for solutions of systems of linear equations will be used in the proof of our first stability Theorem 4.2 in the next section.

Proposition 3.1 *Let $A \in \mathbb{F}^*\mathbb{R}^{m\times n}$ be any matrix such that the standard parts of its rows are linearly independent over $\mathbb{R}$, and let $b \in \mathbb{F}^*\mathbb{R}^m$. Then, for any $x \in \mathbb{F}^*\mathbb{R}^n$ satisfying $A\,x \approx b$, there is $y \in {}^*\mathbb{R}^n$ such that $y \approx x$ and $A\,y = b$.*

Note that the vector y, being infinitesimally close to the finite vector x, is necessarily finite as well.

Proof The assumptions on A imply that its rows are linearly independent over $^*\mathbb{R}$. Indeed, the coefficients of any nontrivial linear dependency over $^*\mathbb{R}$ among the rows of A can be scaled so that all of them are finite and at least one of them is 1; taking standard parts would yield a nontrivial linear dependency over $\mathbb{R}$ among the rows of ${}^\circ\!A$. The assumptions also guarantee that $m \le n$ and both the systems $A\,\xi = b$, ${}^\circ\!A\,\xi = {}^\circ b$ indeed have solutions (in $^*\mathbb{R}^n$, $\mathbb{R}^n$, respectively), because the internal rank of A over $^*\mathbb{R}$, as well as the rank of ${}^\circ\!A$ over $\mathbb{R}$ are both equal to m. We denote by V the orthocomplement of the internal linear subspace $\operatorname{Ker} A = \{\xi \in {}^*\mathbb{R}^n : A\,\xi = 0\}$ in $^*\mathbb{R}^n$.

Let $x \in \mathbb{F}^*\mathbb{R}^n$ satisfy $A\,x \approx b$ and take $y \in {}^*\mathbb{R}^n$ to be the orthogonal projection of x to the affine subspace $\{\xi \in {}^*\mathbb{R}^n : A\,\xi = b\}$ of $^*\mathbb{R}^n$, which means that $A\,y = b$ and $x - y \in V$. It suffices to prove that $x \approx y$.

The product $A\,A^\top$ is a finite, symmetric, positive semi-definite $m \times m$ matrix over $^*\mathbb{R}$. As noted in the first paragraph of Bernstein [3, Section 5.6], this product has rank m over $^*\mathbb{R}$, so 0 is not an eigenvalue of $A\,A^\top$. (See also [3, Theorem 5.6.2(i)].) Let $e_1 \ge \cdots \ge e_m > 0$ be the eigenvalues of $A\,A^\top$, which must all be finite since $A\,A^\top$ is a finite matrix, and let $d_i = \sqrt{e_i}$ for $i = 1, \ldots, m$. Let D be the $m \times n$ matrix with d_i in position (i,i) for $i \le m$ and 0 in all remaining positions. Then [3, Theorem 5.6.3] yields that there exist orthogonal matrices P, Q over $^*\mathbb{R}$ such that $A = PDQ^\top$. (This is the *singular value decomposition* of A; see eg [3, Sections 5.6 and 9.11] and Han and Neumann [9, Section 5.6].) Then, because the standard part map preserves sums, products and the transpose of finite matrices, the matrices ${}^\circ P$, ${}^\circ Q$ are also orthogonal, and ${}^\circ\!A = {}^\circ P\,{}^\circ D\,{}^\circ Q^\top$. Further ${}^\circ e_1, \ldots, {}^\circ e_m$ are the eigenvalues of ${}^\circ\!A\,{}^\circ\!A^T$ and ${}^\circ d_i = \sqrt{{}^\circ e_i}$ for $i = 1, \dots, m$. As ${}^\circ\!D$ is a diagonal matrix with the diagonal formed by the elements ${}^\circ d_1 \ge \dots \ge {}^\circ d_m \ge 0$, we see that ${}^\circ\!A = {}^\circ\!P\,{}^\circ\!D\,{}^\circ Q^\top$ is the singular value decomposition of ${}^\circ\!A$. It follows that ${}^\circ\!D$ has rank m, and hence ${}^\circ d_m > 0$. Since $A\,Q = P\,D$, denoting by $u_1, \dots, u_m$ and $v_1, \dots, v_n$ the columns of the matrices P and Q, respectively, we have $A\,v_i = d_i u_i$ for $i \le m$ and $A\,v_i = 0$ for $m < i \le n$. Thus the columns $v_1, \dots, v_m$ span V while the columns $v_{m+1}, \dots, v_n$ span $\operatorname{Ker} A$. Then, using the fact that the vectors $u_1, \dots, u_m$ and $v_1, \dots, v_n$ form orthonormal bases of the internal inner product spaces ${}^*\mathbb{R}^m$ and ${}^*\mathbb{R}^n$, respectively, we have, for any vector $v = c_1 v_1 + \dots + c_m v_m \in V$,

$$\begin{aligned} \|A\,v\| &= \|c_1\,A\,v_1 + \dots + c_m\,A\,v_m\| = \|c_1 d_1 u_1 + \dots + c_m d_m u_m\| \\ &= \sqrt{c_1^2 d_1^2 + \dots + c_m^2 d_m^2} \ge d_m \sqrt{c_1^2 + \dots + c_m^2} = d_m\,\|v\|\ . \end{aligned}$$

In particular, since $x - y \in V$ and $A\,x \approx b = A\,y$,

$$d_m\,\|x - y\| \le \|A(x - y)\| = \|A\,x - b\| \approx 0\,,$$

implying $\|x - y\| \approx 0$, ie $x \approx y$. □

4 The "almost-near" theorems for dual lattices nonstandard formulation

Given an internal lattice $L = \prod L_i \big/ D$ in ${}^*\mathbb{R}^n$, its *internal integral annihilator* can be defined as the ultraproduct of the integral annihilators of the particular lattices $L_i \subseteq \mathbb{R}^n$ or, equivalently, as the annihilator of L with respect to the set of hyperintegers ${}^*\mathbb{Z}$. Then the Łos Theorem (Lemma 2.1) assures that both the objects coincide, ie

$$\operatorname{Ann}_{^*\mathbb{Z}}(L) = \{u \in {}^*\mathbb{R}^n : \forall\, x \in L\colon\, ux \in {}^*\mathbb{Z}\} = \prod \operatorname{Ann}_{\mathbb{Z}}(L_i) \Big/ D\,.$$

Similarly, we have a two-fold definition of the *internal dual* of the internal lattice L:

$$L' = \operatorname{Ann}_{^*\mathbb{Z}}(L) \cap {}^*\mathrm{span}(L) = \prod L_i' \Big/ D$$

Using the transfer principle, Lemma 1.3 implies the following transference relations between the successive minima of an internal lattice $L \subseteq {}^*\mathbb{R}^n$ and the successive minima and the covering radius, respectively, of its internal dual lattice.

Lemma 4.1 *Let $L \subseteq \mathbb{R}^n$ be an internal lattice of rank m. Then*

$$\lambda_k(L)\,\lambda_{m-k+1}\big(L'\big) < \infty \qquad \text{for each } k \le m$$

and

$$\lambda_1(L)\,\mu\big(L'\big) < \infty.$$

Remark 2 The preceding relations follow already from considerably weaker estimates than those in Lemma 1.3, namely $\lambda_k(L)\,\lambda_{m-k+1}\bigl(L'\bigr) \le m!$ (due to Mahler [15]) and the almost obvious observation $\mu\bigl(L'\bigr) \le \frac{1}{2}\,m\,\lambda_m\bigl(L'\bigr)$, which jointly imply:

$$\lambda_1(L)\,\mu\bigl(L'\bigr) \le \frac{1}{2}\,m\,m!$$

Even weaker estimates $\lambda_k(L)\,\lambda_{m-k+1}\bigl(L'\bigr) \le (m!)^2$ (see Gruber and Lekkerkerker [8, page 125]) are still sufficient (cf Remark 1).

As first we prove an infinitesimal version of the "almost-near" result for integral annihilators of internal lattices.

Theorem 4.2 *Let $L \subseteq {}^*\mathbb{R}^n$ be an internal lattice. Then for each $x \in \mathbb{F}^*\mathbb{R}^n$, such that $|u\,x|_{\mathbb{Z}} \approx 0$ for every finite $u \in L$, there is $y \in \mathrm{Ann}_{^*\mathbb{Z}}(L)$ such that $y \approx x$.*

Proof Let $\boldsymbol{\beta} = (v_1, \ldots, v_m)$ be a Minkowski reduced basis of L, and $0 \le p \le q \le m$ be natural numbers such that $v_1, \ldots, v_p$ are all the infinitesimal vectors in $\boldsymbol{\beta}$ and $v_1, \ldots, v_q$ are all the finite vectors in $\boldsymbol{\beta}$. Recalling Proposition 2.4 (b) and its proof, there are hyperintegers $c_1, \ldots, c_p \in {}^*\mathbb{Z}$ such that the vectors $c_1v_1, \ldots, c_pv_p \in L$ are finite and noninfinitesimal and each finite vector $u \in L$ is infinitesimally close to a vector of the form

$$(a_1c_1v_1 + \ldots + a_pc_pv_p) + (a_{p+1}v_{p+1} + \ldots + a_qv_q)\,,$$

where $a_1, \ldots, a_p \in \mathbb{R}$ and $a_{p+1}, \ldots, a_q \in \mathbb{Z}$.

Form the matrix with columns $c_1v_1, \ldots, c_pv_p, v_{p+1}, \ldots, v_q$, and denote by $A \in \mathbb{F}^*\mathbb{R}^{q\times n}$ its transpose. Then $x \in \mathbb{F}^*\mathbb{R}^n$ satisfies the condition $|u\,x|_{\mathbb{Z}} \approx 0$ for each finite $u \in L$ if and only if $c_kv_k\,x \approx 0$ for $k \le p$ and $|v_k\,x|_{\mathbb{Z}} \approx 0$ for $p < k \le q$. Assume that u satisfies this condition and put $b = \bigl(0, \ldots, 0, {}^\circ(v_{p+1}x), \ldots, {}^\circ(v_qx)\bigr)^\top$. Then $b \in \mathbb{Z}^q$ and x satisfies $A\,x \approx b$. By virtue of Proposition 3.1, there is $y \in \mathbb{F}^*\mathbb{R}^n$ such that $y \approx x$ and $A\,y = b$. Then, however, $v_k\,y = b_k = 0$ for $k \le p$, and $v_k\,y = b_k \in \mathbb{Z}$ for $p < k \le q$. If $q = m$, we are done. Otherwise there exists a sequence of integers $q = q_0 < q_1 < \ldots < q_t = m$, such that

$$\|v_{q_{s-1}}\| \ll \|v_k\| \sim \|v_{q_s}\|$$

for all s, k satisfying $1 \le s \le t$, $q_{s-1} < k \le q_s$.

We are going to construct a sequence of vectors $y^{(0)} = y$, $y^{(1)}, \ldots, y^{(t)} \in \mathbb{F}^*\mathbb{R}^n$, such that $y^{(s)} \approx x$ and $v_k\,y^{(s)} \in {}^*\mathbb{Z}$ for any $s \le t$, $k \le q_s$. Then $v\,y^{(t)} \in {}^*\mathbb{Z}$ for every $v \in L$, as required. This will be achieved by an inductive argument. Obviously it is enough to prove the following:

Claim *Let $0 \leq s < t$ and $z \in \mathbb{F}{}^*\mathbb{R}^n$ be a vector such that $v_k z \in {}^*\mathbb{Z}$ for any $k \leq q_s$. Then there is $z' \in \mathbb{F}{}^*\mathbb{R}^n$ such that $z' \approx z$ and $v_k z' \in {}^*\mathbb{Z}$ for any $k \leq q_{s+1}$.*

Let us denote $q' = q_s$, $q'' = q_{s+1}$, $d = q'' - q' > 0$, and form the internal lattice $M = {}^*\mathrm{grp}(v_1, \dots, v_{q''}) \subseteq L$, as well as the internal linear subspace $V = {}^*\mathrm{span}(M) = {}^*\mathrm{span}(v_1, \dots, v_{q''}) \subseteq {}^*\mathbb{R}^n$. According to Lemma 2.2 (b) and Lemma 4.1 we know that

$$\|v_k\| \sim \lambda_k(M) \quad \text{and} \quad \lambda_k(M)\, \lambda_{q''-k+1}\big(M'\big) < \infty$$

whenever $q' < k \leq q''$. Putting both the relations together, for $k = q' + 1$ we get

$$\big\|v_{q'+1}\big\|\ \lambda_d\big(M'\big) < \infty\,.$$

Since the vectors v_k, for $q < k \leq m$, are infinite, we see that $\lambda_d\big(M'\big) \approx 0$. Thus there are vectors $w_1, \dots, w_d \in M'$, linearly independent over ${}^*\mathbb{R}$ such that $\|w_j\| \leq \lambda_d\big(M'\big)$ for $j \leq d$; in particular, all the vectors w_j are infinitesimal.

We will search for the vector z' in the form

$$z' = z + \alpha_1 w_1 + \ldots + \alpha_d w_d$$

with unknown coefficients $\alpha_1, \dots, \alpha_d \in \mathbb{F}{}^*\mathbb{R}$. This will guarantee that $z' \approx z$.

As $\|v_k\| \ll \big\|v_{q'+1}\big\|$, for any $k \leq q'$, $j \leq d$, we have $\|v_k\| \ll \big\|v_{q'+1}\big\|$ and

$$|v_k\, w_j| \leq \|v_k\|\, \|w_j\| \leq \frac{\|v_k\|}{\big\|v_{q'+1}\big\|}\ \big\|v_{q'+1}\big\|\ \lambda_d\big(M'\big) \approx 0\,.$$

At the same time, $v_k\, w_j \in {}^*\mathbb{Z}$, hence $v_k\, w_j = 0$, and

$$v_k\, z' = v_k\, z + \sum_{j=1}^{d} \alpha_j v_k\, w_j = v_k\, z \in {}^*\mathbb{Z}\,,$$

regardless of the choice of $\alpha_1, \dots \alpha_d$. Moreover, denoting by $h\colon {}^*\mathbb{R}^n \to {}^*\mathbb{R}^d$ the ${}^*\mathbb{R}$-linear mapping given by $h(\xi) = (\xi\, w_1, \dots, \xi\, w_d)^\top$ for $\xi \in {}^*\mathbb{R}^n$, we can conclude that the vectors $v_1, \dots, v_{q'}$ form a basis of the linear subspace $V \cap \mathrm{Ker}\, h \subseteq {}^*\mathbb{R}^n$. Indeed, as the vectors $w_1, \dots, w_d$ are linearly independent, $\mathrm{Ker}\, h$ has dimension $n - d$ and it equals the direct sum of the orthocomplement $V^\perp$ with dimension $n - q''$ and $V \cap \mathrm{Ker}\, h$. Then the latter necessarily has dimension $(n - d) - (n - q'') = q'$.

On the other hand, for $q' < k \leq q''$, $j \leq d$, we still have $\|v_k\| \sim \big\|v_{q'+1}\big\|$ and

$$|v_k\, w_j| \leq \|v_k\|\, \|w_j\| \leq \frac{\|v_k\|}{\big\|v_{q'+1}\big\|}\ \big\|v_{q'+1}\big\|\ \lambda_d\big(M'\big) < \infty\,,$$

hence each $v_k\, w_j$ is a finite integer, and $h(v_k) \in \mathbb{Z}^d$ for any k. Since the vectors $v_1, \dots, v_{q'}, v_{q'+1}, \dots, v_{q''}$ are linearly independent over ${}^*\mathbb{R}$ and the first q' from among them form a basis of $V \cap \operatorname{Ker}\psi$, the vectors $h(v_{q'+1}), \dots, h(v_{q''})$ are linearly independent over ${}^*\mathbb{R}$ as well. Then the matrix $B = (b_{ij}) \in {}^*\mathbb{R}^{d\times d}$ with entries $b_{ij} = v_{q'+i}\, w_j \in \mathbb{Z}$ satisfies $0 \not\approx \det B \in \mathbb{Z}$. It follows that B is strongly regular and B^{-1} is finite. Thus denoting by $\omega = (\omega_1, \dots, \omega_d)^\top \in \mathbb{F}^*\mathbb{R}^d$ the vector with coordinates $\omega_j = v_{q'+i}\, z - \lfloor v_{q'+i}\, z \rfloor$ (ie, the fractional parts of the inner products $v_{q'+i}\, z$), for $i \le d$, the system $B\,\eta = -\omega$ has a unique solution $\alpha = (\alpha_1, \dots, \alpha_d)^\top = -B^{-1}\,\omega \in \mathbb{F}^*\mathbb{R}^d$, which means that

$$\sum_{j=1}^{d} v_{q'+i}\, w_j\, \alpha_j = -\omega_i$$

for each $i \le d$. Taking any $q' < k \le q''$ and putting $i = k - q'$, the following computation

$$\begin{aligned} v_k\, z' &= v_k\, z + \sum_{j=1}^{d} \alpha_j v_{q'+i}\, w_j = v_k\, z + \sum_{j=1}^{d} b_{ij}\alpha_j \\ &= v_{q'+i}\, z - \omega_i = \lfloor v_{q'+i}\, z \rfloor \in {}^*\mathbb{Z} \end{aligned}$$

concludes the proof of the claim, hence of the theorem. □

Corollary 4.3 *Let $L \subseteq {}^*\mathbb{R}^n$ be an internal lattice. Then:*

$$^\circ(\operatorname{Ann}_{^*\mathbb{Z}} L) = \operatorname{Ann}_{\mathbb{Z}}\big({}^\circ L\big)$$

In other words, the standard part of the internal integral annihilator $\operatorname{Ann}_{^*\mathbb{Z}} L$ of L equals the integral annihilator of the standard part ${}^\circ L$ of L.

Proof The inclusion $\operatorname{Ann}_{\mathbb{Z}}({}^\circ L) \subseteq {}^\circ(\operatorname{Ann}_{^*\mathbb{Z}} L)$ is a direct consequence of the last theorem. Indeed, if $x \in \operatorname{Ann}_{\mathbb{Z}}({}^\circ L)$ then $x\,{}^\circ u \in \mathbb{Z}$ for every finite $u \in L$. Then $|x\,u|_{\mathbb{Z}} \approx 0$ for any such u, and by Theorem 4.2 there is $y \in \operatorname{Ann}_{^*\mathbb{Z}}(L)$ such that $y \approx x$, hence $x \in {}^\circ(\operatorname{Ann}_{^*\mathbb{Z}} L)$.

The reversed inclusion ${}^\circ(\operatorname{Ann}_{^*\mathbb{Z}} L) \subseteq \operatorname{Ann}_{\mathbb{Z}}({}^\circ L)$ is easy. It suffices to show that ${}^\circ x \in \operatorname{Ann}_{\mathbb{Z}}\big({}^\circ L\big)$ for each finite $x \in \operatorname{Ann}_{^*\mathbb{Z}}(L)$. Taking any finite $u \in L$, the inner product $u\,x$ is finite and belongs to ${}^*\mathbb{Z}$, hence

$${}^\circ u\, {}^\circ x = {}^\circ(u\,x) = u\,x \in \mathbb{Z}\,,$$

so that ${}^\circ x \in \operatorname{Ann}_{\mathbb{Z}}\big({}^\circ L\big)$ as required. □

The following is the nonstandard formulation of the “almost-near” result for dual lattices.

Theorem 4.4 *Let $L \subseteq {}^*\mathbb{R}^n$ be an internal lattice. Then for each finite vector $x \in {}^*\mathrm{span}(L)$, such that $|u\,x|_{\mathbb{Z}} \approx 0$ for every finite $u \in L$, there is $y \in L'$ such that $y \approx x$.*

Proof Let $V = {}^*\mathrm{span}(L) \subseteq {}^*\mathbb{R}^n$ and z_V denote the orthogonal projection of any $z \in {}^*\mathbb{R}^n$ to V. Then $\|z_V\| \le \|z\|$ for any z. According to Theorem 4.2, under the above assumptions there is $y \in \mathrm{Ann}_{{}^*\mathbb{Z}}(L)$ such that $y \approx x$. Then $v\,y_V = v\,y \in {}^*\mathbb{Z}$ for every $v \in L$, ie $y_V \in L'$. As $x_V = x$ and $z \mapsto z_V$ is a linear map,

$$\|x - y_V\| = \|x_V - y_V\| = \|(x-y)_V\| \le \|x - y\| \approx 0\,,$$

hence $y_V \approx x$. □

The last stability theorem is equivalent to the inclusion $\left({}^\circ L\right)' \subseteq {}^\circ\left(L'\right)$ for internal lattices $L \subseteq {}^*\mathbb{R}^n$. In view of Corollary 4.3 the reader might expect the reversed inclusion ${}^\circ\left(L'\right) \subseteq \left({}^\circ L\right)'$ to be satisfied (and even easy to prove). However, as shown by the following example, this is not true in general.

Example 4.5 Let $c \in \mathbb{R}$ be positive and $d \in {}^*\mathbb{R}$ be positive and infinite. Consider the full rank internal lattice

$$L = c\,{}^*\mathbb{Z} \times d\,{}^*\mathbb{Z} = \{(ac, bd)^\top : a, b \in {}^*\mathbb{Z}\}$$

in ${}^*\mathbb{R}^2$. Then, as easily seen, its standard part is a rank 1 lattice ${}^\circ L = c\,\mathbb{Z} \times \{0\}$ in $\mathbb{R}^2$, while its internal dual is the full rank internal lattice $L' = c^{-1}{}^*\mathbb{Z} \times d^{-1}{}^*\mathbb{Z}$ in ${}^*\mathbb{R}^2$. Then $\left({}^\circ L\right)' = c^{-1}\mathbb{Z} \times \{0\}$ is a rank 1 lattice in $\mathbb{R}^2$ while ${}^\circ\left(L'\right) = c^{-1}\mathbb{Z} \times \mathbb{R}$ is not even a lattice.

5 The "almost-near" theorem for dual lattices standard formulation

Here we prove Theorem 5.2, which is our main stability result for dual lattices. It is the standard equivalent of Theorem 4.4. In its proof we will need the following nonstandard lemma.

Lemma 5.1 *Let $L \subseteq {}^*\mathbb{R}^n$ be an internal lattice and $G \subseteq L$ be any additive subgroup of L. Further, let $\delta < \frac{1}{3}$ be a positive real number and $x \in {}^*\mathbb{R}^n$ be a vector such that $|u\,x|_{{}^*\mathbb{Z}} \le \delta$ for every $u \in G$. Then $|u\,x|_{{}^*\mathbb{Z}} \approx 0$ for every $u \in G$.*

Proof As the mapping $u \mapsto u\,x$ is an additive group homomorphism $L \to {}^*\mathbb{R}$, the image $G\,x = \{u\,x : u \in G\}$ of the subgroup $G \subseteq L$ under this map must be a subgroup of ${}^*\mathbb{R}$. However, as $0 < \delta < \frac{1}{3}$ is a (standard) real number, ${}^*\mathbb{Z} + \mathbb{I}^*\mathbb{R}$ is the biggest subgroup of ${}^*\mathbb{R}$ satisfying the inclusion ${}^*\mathbb{Z} + \mathbb{I}^*\mathbb{R} \subseteq {}^*\mathbb{Z} + {}^*[-\delta, \delta]$. $\square$

Recall that $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denotes the (Euclidian) unit ball in $\mathbb{R}^n$.

Theorem 5.2 *Let $n \geq 1$ be an integer and $\delta < \frac{1}{3}$, ε, λ be positive reals. Then there exists a real number $r > 0$, depending just on n, δ, ε and λ, such that every lattice $L \subseteq \mathbb{R}^n$, subject to $\lambda_1(L) \geq \lambda$, satisfies the following condition:*

> *For any $x \in \operatorname{span}(L)$ such that $|u\,x|_{\mathbb{Z}} \leq \delta$ for all $u \in L \cap rB$, there is $y \in L'$ such that $\|x - y\| \leq \varepsilon$.*

Proof Assume that the conclusion of the theorem fails for some fixed quadruple of admissible parameters n, δ, ε, λ. This is to say that for each real number $r > 0$ there is a lattice $L_r \subseteq \mathbb{R}^n$, satisfying $\lambda_1(L_r) \geq \lambda$, and $x_r \in \operatorname{span}(L)$ such that $|u\,x_r|_{\mathbb{Z}} \leq \delta$ for every $u \in L_r \cap rB$, however $\|x_r - y\| > \varepsilon$ for any $y \in L'$; ie, $(x_r + \varepsilon B) \cap L_r' = \emptyset$. In particular, this is true for values of r from the set $I = \{1, 2, 3, \dots\}$ of all positive integers.

Let D be any nonprincipal ultrafilter on the set I. Form the ultraproduct $L = \prod_{r \in I} L_r \big/ D$, the vector $x = (x_r)_{r \in I} \big/ D \in L$, and the infinite positive hyperinteger $\rho = (1, 2, 3, \dots)/D$. Then, by the virtue of the Łos Theorem (Lemma 2.1), $L \subseteq {}^*\mathbb{R}^n$ is an internal lattice satisfying $\lambda_1(L) \geq \lambda$. For the same reason we have (i) $x \in {}^*\operatorname{span}(L)$, (ii) $|u\,x|_{{}^*\mathbb{Z}} \leq \delta$ for every $u \in L \cap \rho\,{}^*B$, and (iii) $(x + \varepsilon\,{}^*B) \cap L' = \emptyset$. Because $\mathbb{F}L = L \cap \mathbb{F}^*\mathbb{R}^n$ is a subgroup of L and $\mathbb{F}L \subseteq L \cap \rho\,{}^*B$, Lemma 5.1 together with (ii) imply that $|u\,x|_{{}^*\mathbb{Z}} \approx 0$ for every $u \in \mathbb{F}L$.

As a consequence of Lemma 4.1, the covering radius $\mu = \mu\big(L'\big)$ is a finite positive hyperreal. (In fact, by Lemma 1.3 and the Łos Theorem, $\mu \leq n^{3/2}/(2\lambda)$.) Thus there is $z \in L'$ such that $\|x - z\| \leq \mu$. Then $x - z \in {}^*\operatorname{span}(L)$ and $u\,(x - z) - u\,x = -u\,z \in {}^*\mathbb{Z}$, hence $|u\,(x - z)|_{{}^*\mathbb{Z}} = |u\,x|_{{}^*\mathbb{Z}}$ for any $u \in L$. At the same time,

$$(x - z + \varepsilon\,{}^*B) \cap L' = (x + \varepsilon\,{}^*B) \cap L' = \emptyset\,.$$

We can conclude, that $x' = x - z \in {}^*\operatorname{span}(L)$ is a finite vector satisfying $|u\,x'|_{{}^*\mathbb{Z}} \approx 0$ for every finite $u \in L$, and $\|x' - y\| > \varepsilon$ for any $y \in L'$. This, however, contradicts Theorem 4.4. $\square$

Final remark Theorems 4.2 and 4.4 are robust in the sense that they do not explicitly involve any norm on $\mathbb{R}^n$ in their formulation. Moreover,

$$\mathbb{F}^*\mathbb{R}^n = \{x \in {}^*\mathbb{R}^n : \|x\| < \infty\} \quad \text{and} \quad \mathbb{I}^*\mathbb{R}^n = \{x \in {}^*\mathbb{R}^n : \|x\| \approx 0\}$$

for any norm $\|x\|$ on $\mathbb{R}^n$ and not just for the Euclidian one. As a consequence, Theorem 5.2 remains true even if B denotes any centrally symmetric convex body in $\mathbb{R}^n$, $\lambda_1(L)$ is replaced by the first Minkowski successive minimum

$$\lambda_1(C, L) = \min\{s \in \mathbb{R} : s > 0,\ L \cap s\,C \neq \{0\}\}$$

of L with respect to another centrally symmetric convex body $C \subseteq \mathbb{R}^n$, and $\|x\|$ is an arbitrary norm on $\mathbb{R}^n$ (possibly without any direct relation either to B or to C).

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