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On Preparation Theorems for $\mathbb{R}_{\text{an,exp}}$–definable Functions

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Abstract: In this article we give strong versions for preparation theorems for $\mathbb{R}_{\text{an,exp}}$–definable functions outgoing from methods of Lion and Rolin. ($\mathbb{R}_{\text{an,exp}}$ is the o–minimal structure generated by all restricted analytic functions and the global exponential function.) By a deep model theoretic fact of van den Dries, Macintyre and Marker every $\mathbb{R}_{\text{an,exp}}$–definable function is piecewise given by $\mathcal{L}_{\text{an}}(\exp, \log)$–terms where $\mathcal{L}_{\text{an}}(\exp, \log)$ denotes the language of ordered rings augmented by all restricted analytic functions, the global exponential and the global logarithm. The idea is to consider log-analytic functions at first, ie functions which are iterated compositions from either side of globally subanalytic functions and the global logarithm, and then $\mathbb{R}_{\text{an,exp}}$–definable functions as compositions of log-analytic functions and the global exponential.

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Introduction

This paper contributes to analysis in the framework of o–minimal structures. O–minimality is a concept from mathematical logic with connections and applications to geometry, analysis, number theory and other areas. Sets and functions definable in an o–minimal structure (ie 'belonging to') exhibit tame geometric and combinatorial behaviour. We refer to the book of van den Dries [2] for the general properties of o–minimal structures; in the preliminary section we state the definition and give examples.

Log-analytic functions have been defined by Lion and Rolin in their seminal paper [9] (see also Kaiser–Opris [8] for the definition). They are iterated compositions from either side of globally subanalytic functions (see van den Dries and Miller [5]) and the global logarithm.

Example 0.1 *The function*

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (t, x) \mapsto \arctan\left(\log\left(\max\{\log(t^4 + \log(x^2 + 2)), 1\}\right)\right)$$

is log-analytic.

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Their definition is kind of hybrid, as we will explain. The globally subanalytic functions are precisely the functions that are definable in the o–minimal structure $\mathbb{R}_{\text{an}}$ of restricted analytic functions (see [5]). Because the global logarithm is not globally subanalytic the class of log-analytic functions contains properly the class of $\mathbb{R}_{\text{an}}$–definable functions. Since from the global logarithm the exponential function is definable, and the latter is not log-analytic, the class of log-analytic functions is not a class of definable functions. It is properly contained in the class of $\mathbb{R}_{\text{an,exp}}$–definable functions. ($\mathbb{R}_{\text{an,exp}}$ is the o–minimal structure generated by all restricted analytic functions and the global exponential function; see [5].) Hence log-analytic functions capture $\mathbb{R}_{\text{an}}$–definability but not full $\mathbb{R}_{\text{an,exp}}$–definability. To obtain full $\mathbb{R}_{\text{an,exp}}$–definability one has to consider compositions of log-analytic functions and the global exponential (see van den Dries, Macintyre and Marker [3]).

In light of these considerations Lion and Rolin found a way to prove that $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{an,exp}}$ have quantifier elimination without using model theoretical techniques (see [9]). The o–minimality of these structures follow from this quantifier elimination result. Their main idea was to introduce and prove preparation theorems: once a definable function $f(t,x)$ is cellwise prepared with respect to a given variable x, it is much easier to solve the equation $f(t,x)=0$ with respect to the unknown x. In some sense, these statements can be understood as a monomialization of the definable functions with respect to x, in the spirit of resolution of singularities for functions which involve the logarithm and exponential functions. There are two mistakes in [9]. The first one, which concerns the log-analytic statement, has been corrected by Pawłucki and Piękosz [13]. The second one, which concerns the proof of the log-exp-analytic statement, has been fixed by van den Dries and Speissegger [6], in the framework of the exponential closure of polynomially bounded o–minimal structures (on the reals).

These preparation theorems are crucial to deal with questions from analysis in the o–minimal context: van den Dries and Miller proved a parametric version of Tamm's Theorem for globally subanalytic functions (see [4]) and Tobias Kaiser showed the impressive fact that a real analytic globally subanalytic function has a holomorphic extension which is again globally subanalytic (see [7]). But it is not straightforward to generalize these results on $\mathbb{R}_{\text{an,exp}}$–definable functions: an easy counterexample shows that Tamm's Theorem does not hold in this setting and the second question is completely open in this framework. To obtain a positive answer for the second issue one idea would be to look closer at preparing $\mathbb{R}_{\text{an,exp}}$–definable functions.

For the rest of the introduction "definable" means "$\mathbb{R}_{\text{an,exp}}$–definable" .

For example a log-analytic function $f(t,x)$ where x is the last variable can be cellwise prepared as $f(t,x)=a(t)|y_0(t,x)|^{q_0}\cdot\ldots\cdot|y_r(t,x)|^{q_r}u(t,x)$ where $y_0(t,x)=x-\Theta_0(t)$ and inductively $y_l(t,x) = \log(|y_{l-1}(t,x)|) - \Theta_l(t)$ for $l \in \{1,\dots,r\}$, the q_j's are rational exponents and $u(t,x)$ is a unit of a special form. This gives roughly that the function $f(t,-)$ behaves cellwise as iterated logarithms independently of t where the order of iteration is bounded in terms of f. (Compare also with van den Dries and Speissegger in [6].) Note that the functions $a(t), \Theta_0(t), \dots, \Theta_r(t)$ although being definable are in general not log-analytic anymore. We will present an example below. But it turns out that a log-analytic function can be prepared with data of a special form.

Given the projection $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, (t,x) \mapsto t$, on the first n coordinates and a definable cell $C \subset \mathbb{R}^n \times \mathbb{R}$, we call a function $g : \pi(C) \to \mathbb{R}$ *C–nice* if g is the composition of log-analytic functions and exponentials of the form $\exp(h)$ where h is the component of a center of a logarithmic scale on C (compare with Definition 2.1 for the definition of a logarithmic scale; the logarithmic scale may depend on g). We call such a function h *C–heir* (see Definition 2.30 below). It is immediately seen that a log-analytic function on $\pi(C)$ is C–nice and that every C–nice function is definable. Note that the class of C–nice functions does not necessarily coincide with the class of definable functions: if the cell C is *simple*, ie for every $t \in \pi(C)$ there is $d_t \in \mathbb{R}_{>0} \cup \{\infty\}$ such that $C_t =]0, d_t[$ (see for example Definition 2.15 in Kaiser–Opris [8]), the class of C–nice functions coincides with the class of log-analytic ones (since in [8] it is shown that the center of a logarithmic scale vanishes on such a cell).

With the class of C–nice functions we are able to prove the following preparation theorem for log-analytic functions. For practical reasons we consider finitely many log-analytic functions and prepare them simultaneously.

Theorem 0.2 *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable, $m \in \mathbb{N}$ and let $f_1, \dots, f_m : X \to \mathbb{R}$ be log-analytic. Then there is $r \in \mathbb{N}_0$ and a decomposition $\mathcal{C}$ of $X_{\neq 0} := \{(t,x) \in X \mid x \neq 0\}$ into finitely many definable cells such that for $C \in \mathcal{C}$ there is $\Theta := (\Theta_0, \dots, \Theta_r)$ such that the functions $f_1|_C, \dots, f_m|_C$ are nicely log-analytically prepared in x with center Θ; ie, $f_j(t,x) = a_j(t)|y_0(t,x)|^{q_0} \cdot \ldots \cdot |y_r(t,x)|^{q_r} u_j(t,x)$ for $(t,x) \in C$ and $j \in \{1,\dots,m\}$ where $y_0(t,x) = x - \Theta_0(t)$ and inductively $y_l(t,x) = \log(|y_{l-1}(t,x)|) - \Theta_l(t)$ for $l \in \{1,\dots,r\}$, $q_0, \dots, q_r \in \mathbb{Q}$, the functions a_j, Θ_j on $\pi(C)$ are C–nice and u_j is a unit of the form $v_j \circ \phi$ where v_j is a real power series which converges on an open neighborhood of $[-1,1]^s$ and $\phi := (\phi_1, \dots, \phi_s)$ with $\phi(C) \subset [-1,1]^s$ and $\phi_i(t) = b_i(t)|y_0(t,x)|^{p_{i0}} \cdot \ldots \cdot |y_r(t,x)|^{p_{ir}}$, where b_i is C–nice and $p_{i0}, \dots p_{ir} \in \mathbb{Q}$.*

This is a more precise preparation theorem than the version from Lion–Rolin [9] since the data of the preparation is C–nice (and not only definable).

For example if C is simple we have that $a, \Theta_0, \ldots, \Theta_r$ and $b_1, \ldots, b_s$ are log-analytic and therefore the preparation keeps log-analyticity (which cannot be obtained by using the preparation theorem from [9]). Consequently if $g(t) := \lim_{x \searrow 0} f(t,x)$ exists for a log-analytic $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, (t,x) \mapsto f(t,x)$, we see that g is also log-analytic (compare with Theorem 3.1 of Kaiser–Opris [8]). In light of this consideration important differentiability properties for log-analytic functions like non-flatness or a parametric version of Tamm's theorem could be established in [8]. (Note also that a special version of Theorem 0.2 has been proven there: this is Theorem 2.30, a preparation theorem for log-analytic functions on simple cells.)

Our second goal for this paper is to establish a preparation theorem for definable functions again by adapting the arguments of Lion–Rolin from [9] by considering definable functions as compositions of log-analytic functions and the global exponential. For this we introduce the concept of the *exponential number* of a definable function f with respect to a set E of positive definable functions (or exponentials), ie the minimal number of iterations of functions from E which are necessary to write f as a composition of log-analytic functions and functions from E.

Example 0.3 *The function* $\mathbb{R} \to \mathbb{R}, x \mapsto \exp(x^2+1)$, *has exponential number at most* 1 *with respect to* $E := \{\exp(x^2+1)\}$. *Since* $\mathbb{R}_{>0} \to \mathbb{R}, y \mapsto \log(y+1)$, *is log-analytic, the function*

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \log(\exp(x^2+1)+1)$$

has also exponential number at most 1 with respect to E. *The function*

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto \exp(\exp(x^2+1))$$

has exponential number at most 2 *with respect to* $\{\exp(x^2+1), \exp(\exp(x^2+1))\}$.

Every function $f : X \to \mathbb{R}$ which has exponential number at most $e \in \mathbb{N}_0$ with respect to a set of positive definable functions on X is definable; and, for every definable function $g : X \to \mathbb{R}$, there is $e \in \mathbb{N}_0$ and a set E of positive definable functions on X such that g has exponential number at most e with respect to E (see Section 1). The notion of the exponential number allows us to prove the following preliminary version of the preparation theorem for definable functions which gives a connection between the functions contained in E and the exponentials which occur in the cellwise preparation of f.

Theorem 0.4 *Let* $e \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$. *Let* $X \subset \mathbb{R}^m$ *be definable and let* $f : X \to \mathbb{R}$ *be a function. Let* E *be a set of positive definable functions on* X *such that* f *has exponential number at most* $e \in \mathbb{N}_0$ *with respect to* E. *Then there is a decomposition*

$\mathcal{D}$ of X into finitely many definable cells such that for every $D \in \mathcal{D}$ we have that $f|_D = a \cdot \exp(c) \cdot u$, where u is a unit of the form $v \circ \phi$ where v is a real power series, and $\phi := (\phi_1, \dots, \phi_s)$ with $\phi_i(t) = b_i(t)\exp(d_i(t))$. Additionally a (respectively b_i) is log-analytic, and c (respectively d_i) is a finite $\mathbb{Q}$–linear combination of functions from $\log(E) := \{\log(g) \mid g \in E\}$ which have exponential number at most $e-1$ with respect to E.

Finally we combine Theorem 0.2 and Theorem 0.4 and prove a preparation theorem for definable functions which is the main result of this paper.

Let $C \subset \mathbb{R}^n \times \mathbb{R}_{\neq 0}$ be a definable cell, let $r \in \mathbb{N}_0$ and let E be a set of positive definable functions on C. Let $\Theta_0,\dots,\Theta_r$ be C–nice. We call a function $f : C \to \mathbb{R}$ $(-1, r)$*–prepared* with center $\Theta := (\Theta_0, \dots, \Theta_r)$ with respect to E if f is the zero function. For $e \in \mathbb{N}_0$ call a function $f : C \to \mathbb{R}$ (e, r)*–prepared* with center Θ with respect to E if for $(t, x) \in C$,

$$f(t,x) = a(t)|y_0(t,x)|^{q_0} \cdot \ldots \cdot |y_r(t,x)|^{q_r} \exp(c(t,x)) \cdot u(t,x)$$

where $q_0, \dots, q_r \in \mathbb{Q}$, $y_0 = x - \Theta_0(t)$, inductively $y_i = \log(|y_{i-1}|) - \Theta_i(t)$ for $i \in \{1, \dots, r\}$, a is C–nice, $\exp(c) \in E$, and the unit u is of the form $u = v \circ \phi$, where v is a real power series with $v(C) \subset [-1, 1]^s$, and $\phi = (\phi_1, \dots, \phi_s)$ with $\phi_i = b_i(t)|y_0(t,x)|^{p_{i0}} \cdot \ldots \cdot |y_r(t,x)|^{p_{ir}} \cdot \exp(d_i(t,x))$ where $\phi_i(C) \subset [-1, 1]$, b_i is C–nice, $p_{i0}, \dots, p_{ir} \in \mathbb{Q}$ and $\exp(d_i) \in E$. Additionally c and d_i are $(e-1, r)$–prepared with center Θ with respect to E.

Theorem 0.5 *Let $e \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let E be a set of positive definable functions on X. Let $f : X \to \mathbb{R}$ be a function with exponential number at most e with respect to E. Then there is $r \in \mathbb{N}_0$ and a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is a finite set P of positive definable functions on C and $\Theta := (\Theta_0, \dots, \Theta_r)$ such that the function $f|_C$ is (e, r)–prepared with center Θ with respect to P and the following hold.*

(1) *For every $g \in \log(P)$ there is $m \in \{-1, \dots, e-1\}$ such that g is (m, r)–prepared in x with center Θ with respect to P.*
(2) *If $g \in \log(P) := \{\log(h) \mid h \in P\}$ is (l, r)–prepared in x with center Θ with respect to P for $l \in \{-1, \dots, e-1\}$ then g is a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$ restricted to C which have exponential number at most l with respect to E.*

Theorem 0.5 gives a cellwise preparation of definable functions in terms of the exponential number and with a precise statement how the unit looks like (compare with [9, Section 2] for a version of Theorem 0.5 not in terms of the exponential number and without mentioning any unit u; compare with [6, Section 5] for a version of Theorem 0.5 in terms of the exponential number where a unit u is mentioned, but without any precise statement how this unit looks like.) In contrast to the treatment in [6] or [9] there is also precise information about the exponentials which occur in the preparation of f on every cell C: they are of the form $\exp(h)$ where h is the component of a center of a logarithmic scale on C (coming from Theorem 0.2, the log-analytic preparation) or h is itself prepared and a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$ restricted to C (coming from Theorem 0.4).

All these improvements together allow us to investigate natural intermediate classes between log-analytic and definable functions and discuss various analytic properties of them. One example are the so-called *restricted log-exp-analytic functions*. These are definable functions which are compositions of log-analytic functions and exponentials of locally bounded functions (see Opris [12, Definition 2.5]). The global exponential function for example is restricted log-exp-analytic. Outgoing from Theorem 0.5 the differentiability results for log-analytic functions from [8] could be generalized by Opris to this class of functions (see [12]). Another consequence of Theorem 0.5 is the deep technical result that a real analytic restricted log-exp-analytic function has a holomorphic extension which is again restricted log-exp-analytic which could also be shown by Opris in [11].

This paper is organised as follows. After a preliminary section about o–minimality, notations and conventions we give in Section 1 the definition of log-analytic functions, the definition of the exponential number of a definable function with respect to a set of positive definable functions and give elementary properties. In Section 2 we present the preparation theorem of Lion–Rolin from [9] for log-analytic functions, introduce 'nice functions' and give a proof for Theorem 0.2 above. Section 3 is devoted to the proof of Theorem 0.4 and Theorem 0.5.

Preliminaries

O–minimality

We give the definition and examples of o–minimal structures.

Semialgebraic sets:
A subset A of $\mathbb{R}^n$, $n \geq 1$, is called *semialgebraic* if there are $k, l \in \mathbb{N}_0$ and real

polynomials $f_i, g_{i,1}, \ldots, g_{i,k} \in \mathbb{R}[X_1, \ldots, X_n]$ for $1 \leq i \leq l$ such that

$$A = \bigcup_{i=1}^{l} \{x \in \mathbb{R}^n \mid f_i(x) = 0, g_{i,1}(x) > 0, \ldots, g_{i,k}(x) > 0\}.$$

A map is called semialgebraic if its graph is semialgebraic.

Semi- and subanalytic sets:
A subset A of $\mathbb{R}^n$, $n \geq 1$, is called *semianalytic* if for every $a \in \mathbb{R}^n$ there are open neighborhoods U, V of a with $\overline{U} \subset V$, $k, l \in \mathbb{N}_0$ and real analytic functions $f_i, g_{i,1}, \ldots, g_{i,k}$ on V for $1 \leq i \leq l$, such that

$$A \cap U = \bigcup_{i=1}^{l} \{x \in U \mid f_i(x) = 0, g_{i,1}(x) > 0, \ldots, g_{i,k}(x) > 0\}.$$

A subset B of $\mathbb{R}^n$, $n \geq 1$, is called *subanalytic* if for every $a \in \mathbb{R}^n$ there is an open neighborhood U of a, some $p \geq n$, and some bounded semianalytic set $A \subset \mathbb{R}^p$ such that $B \cap U = \pi_n(A)$, where $\pi_n : \mathbb{R}^p \to \mathbb{R}^n, (x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_n)$, is the projection on the first n coordinates. A map is called semianalytic or subanalytic if its graph is a semianalytic or subanalytic set, respectively. A set is called *globally semianalytic* or *globally subanalytic* if it is semianalytic or subanalytic, respectively, in the ambient projective space (or equivalently, after applying the semialgebraic homeomorphism $\mathbb{R}^n \to]-1, 1[^n, x_i \mapsto x_i/\sqrt{1 + x_i^2}$.)

O–minimal structures:
A *structure* on $\mathbb{R}$ is axiomatically defined as follows. For $n \in \mathbb{N}$ let M_n be a set of subsets of $\mathbb{R}^n$ and let $\mathcal{M} := (M_n)_{n \in \mathbb{N}}$. Then $\mathcal{M}$ is a structure on $\mathbb{R}$ if the following hold for all $m, n, p \in \mathbb{N}$:

(S1) If $A, B \in \mathbb{R}^n$ then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A \in M_n$. (So M_n is a Boolean algebra of subsets of M_n.)

(S2) If $A \in M_n$ and $B \in M_m$ then $A \times B \in M_{n+m}$.

(S3) If $A \in M_p$ and $p \geq n$ then $\pi_n(A) \in M_n$ where $\pi_n : \mathbb{R}^p \to \mathbb{R}^n, (x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_n)$, denotes the projection on the first n coordinates.

(S4) M_n contains the semialgebraic subsets of $\mathbb{R}^n$.

The structure $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ on $\mathbb{R}$ is called *o–minimal* if additionally the following holds.

(O) The sets in M_1 are exactly the finite unions of intervals and points.

A subset of $\mathbb{R}^n$ is called *definable* in the structure $\mathcal{M}$ if it belongs to M_n. A function is definable in $\mathcal{M}$ if its graph is definable in $\mathcal{M}$. The o–minimality axiom (O) implies that a subset of $\mathbb{R}$ which is definable in an o–minimal structure on $\mathbb{R}$ has only finitely many connected components. But much more can be deduced from o–minimality. A definable subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ arbitrary, has only finitely many connected components and these are again definable. More generally, sets and functions definable in an o–minimal structure exhibit tame geometric behaviour, for example the existence of definable cell decomposition. We refer to the book of van den Dries [2] for this and more of the general properties of o–minimal structures.

Examples of o–minimal structures:

(1) The smallest o–minimal structure on $\mathbb{R}$ is given by the semialgebraic sets. It is denoted by $\mathbb{R}$.
(2) $\mathbb{R}_{\text{exp}}$, the structure generated on the real field by the global exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ (ie the smallest structure containing the semialgebraic sets and the graph of the global exponential function), is o–minimal.
(3) $\mathbb{R}_{\text{an}}$, the structure generated on the real field by the restricted analytic functions, is o–minimal. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called restricted analytic if there is a function g that is real analytic on a neighborhood of $[-1,1]^n$ such that $f = g$ on $[-1,1]^n$ and $f = 0$ otherwise. The sets definable in $\mathbb{R}_{\text{an}}$ are precisely the globally subanalytic ones.
(4) $\mathbb{R}_{\text{an,exp}}$, the structure generated by $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$, is o–minimal.

Notation

The empty sum is by definition 0 and the empty product is by definition 1. By $\mathbb{N} = \{1, 2, \ldots\}$ we denote the set of natural numbers and by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of nonnegative integers. We set $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$. For $m, n \in \mathbb{N}$ we denote by $M(m \times n, \mathbb{Q})$ the set of $m \times n$-matrices with rational entries. Given $x \in \mathbb{R}$ let $\text{sign}(x) \in \{\pm 1\}$ be its sign. By $\log_k$ we denote the k–times iteration of the natural logarithm. By $\sim$ we denote asymptotic equivalence.

For $m \in \mathbb{N}$ and a set $X \subset \mathbb{R}^m$ we set the following. For a set E of positive real valued functions on X we set $\log(E) := \{\log(g) \mid g \in E\}$. For $C \subset \mathbb{R}^m$ with $C \subset X$ and a set E of real valued functions on X we set $E|_C := \{g|_C \mid g \in E\}$.

For $X \subset \mathbb{R}^n \times \mathbb{R}$ let $X_{\neq 0} := \{(t,x) \in X \mid x \neq 0\}$ and for $t \in \mathbb{R}^n$ we set $X_t := \{x \in \mathbb{R} \mid (t,x) \in X\}$.

Convention

Definable means definable in $\mathbb{R}_{\text{an,exp}}$ if not otherwise mentioned.

1 Log-analytic functions and the exponential number

We fix $m \in \mathbb{N}$ and a definable $X \subset \mathbb{R}^m$.

Definition 1.1 *Let $f : X \to \mathbb{R}$ be a function.*

(a) *Let $r \in \mathbb{N}_0$. By induction on r we define that f is* ***log-analytic of order at most*** *r:*
Base case. *The function f is log-analytic of order at most 0 if f is piecewise the restriction of globally subanalytic functions, ie there is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for $C \in \mathcal{C}$ there is a globally subanalytic function $F : \mathbb{R}^m \to \mathbb{R}$ such that $f|_C = F|_C$.*
Inductive step. *The function f is log-analytic of order at most r if the following holds: there is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, a globally subanalytic function $F : \mathbb{R}^{k+l} \to \mathbb{R}$, and log-analytic functions $g_1, \ldots, g_k : C \to \mathbb{R}, h_1, \ldots, h_l : C \to \mathbb{R}_{>0}$ of order at most $r-1$ such that $f|_C = F(g_1, \ldots, g_k, \log(h_1), \ldots, \log(h_l))$.*

(b) *Let $r \in \mathbb{N}_0$. We call f* ***log-analytic of order*** *r if f is log-analytic of order at most r but not of order at most $r-1$.*

(c) *We call f* ***log-analytic*** *if f is log-analytic of order r for some $r \in \mathbb{N}_0$.*

Remark 1.2 *The following properties hold.*

(1) *A log-analytic function is definable.*

(2) *The log-analytic functions are precisely those definable functions which are piecewise given by $\mathcal{L}_{\mathrm{an}}(^{-1}, (\sqrt[n]{\ldots})_{n=2,3,\ldots}, \log)$–terms (compare with van den Dries, Macintyre and Marker in [3]).*

(3) *A function is log-analytic of order 0 if and only if it is piecewise the restriction of globally subanalytic functions.*

Example 1.3 *The function*

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \arctan(\log(\max\{\log(x^4 + \log(y^2 + 2)), 1\}))$$

is log-analytic of order (at most) 3.

Remark 1.4 *Let $r \in \mathbb{N}_0$. The set of log-analytic functions on X of order at most r as well as the set of log-analytic functions on X is an $\mathbb{R}$–algebra with respect to pointwise addition and multiplication.*

Now we consider a definable function f as a composition of log-analytic functions and exponentials. To formalize the iterations of the exponentials we introduce the *exponential number* of f with respect to a set E of positive definable functions: this is the minimal number of iterations of exponentials from E which are necessary to write f as a composition of log-analytic functions and functions from E. (See also van den Dries and Speissegger [6] for such considerations. There the exponential level of a definable function is introduced to describe iterations of exponentials, but without mentioning any set E of positive definable functions.) This allows us to define and study several different classes of definable functions like *C–nice functions* (where E consists of C–heirs, see Section 2.3) or *restricted log-exp-analytic functions* (where E consists of exponentials of locally bounded functions, see Opris [12]). The former helps to understand cellwise preparation of log-analytic functions (see Theorem 0.2) and the latter is a large class of definable functions which contains the log-analytic ones properly and shares some analytic properties with globally subanalytic functions (see Opris [11] and [12]).

Definition 1.5 *Let $f : X \to \mathbb{R}$ be a function. Let E be a set of positive definable functions on X.*

(a) *By induction on $e \in \mathbb{N}_0$ we define that f has **exponential number at most e with respect to E**:*
Base Case. *The function f has exponential number at most 0 with respect to E if f is log-analytic.*
Inductive Step. *The function f has exponential number at most e with respect to E if the following holds: there are $k, l \in \mathbb{N}_0$, functions $g_1, \ldots, g_k : X \to \mathbb{R}$ and $h_1, \ldots, h_l : X \to \mathbb{R}$ with exponential number at most $e-1$ with respect to E and a log-analytic function $F : \mathbb{R}^{k+l} \to \mathbb{R}$ such that*
$$f = F(g_1, \ldots, g_k, \exp(h_1), \ldots, \exp(h_l))$$
and $\exp(h_1), \ldots, \exp(h_l) \in E$.

(b) *Let $e \in \mathbb{N}_0$. We say that f has **exponential number e with respect to E** if f has exponential number at most e with respect to E but not at most $e-1$ with respect to E.*

(c) *We say that f **can be constructed from** E if there is $e \in \mathbb{N}_0$ such that f has exponential number e with respect to E.*

Remark 1.6 *Let E be a set of positive definable functions on X. Let f be a function on X which can be constructed from E. The following hold:*

(1) *f has a unique exponential number $e \in \mathbb{N}_0$ with respect to E and is definable.*
(2) *f has exponential number 0 with respect to E if and only if f is log-analytic.*

Remark 1.7 *Let E be the set of all positive definable functions on X. Then every definable function $f : X \to \mathbb{R}$ can be constructed from E.*

Remark 1.8 *Let $e \in \mathbb{N}_0$. Let E be a set of positive definable functions on X.*

(1) *Let $f : X \to \mathbb{R}$ be a function with exponential number at most e with respect to E. Then $\exp(f)$ has exponential number at most $e+1$ with respect to $E \cup \{\exp(f)\}$.*
(2) *Let $s \in \mathbb{N}_0$. Let $f_1, \ldots, f_s : X \to \mathbb{R}$ be functions with exponential number at most e with respect to E and let $F : \mathbb{R}^s \to \mathbb{R}$ be log-analytic. Then $F(f_1, \ldots, f_s)$ has exponential number at most e with respect to E.*

Proof (1) One sees with Definition 1.5 applied to $g := F(\exp(f))$ where $F = \mathrm{id}_{\mathbb{R}}$ that $\exp(f)$ has exponential number at most $e+1$ with respect to $E \cup \{\exp(f)\}$.

(2) We may assume $e > 0$. Let $k, l \in \mathbb{N}_0$, $g_1, \ldots, g_k, h_1, \ldots, h_l : X \to \mathbb{R}$ be functions with exponential number at most $e-1$ with respect to E with $\exp(h_1), \ldots, \exp(h_l) \in E$, and $G_j : \mathbb{R}^{k+l} \to \mathbb{R}$ be log-analytic such that $f_j = G_j(\beta)$ for $j \in \{1, \ldots, s\}$ where $\beta := (g_1, \ldots, g_k, \exp(h_1), \ldots, \exp(h_l))$. Let v range over $\mathbb{R}^{k+l}$. Then

$$H : \mathbb{R}^{k+l} \to \mathbb{R}, \quad v \mapsto F(G_1(v), \ldots, G_s(v))$$

is log-analytic such that $H(\beta) = F(f_1, \ldots, f_s)$. □

2 A preparation theorem for log-analytic functions

For the whole section let $n \in \mathbb{N}_0$, $t := (t_1, \ldots, t_n)$ range over $\mathbb{R}^n$, x over $\mathbb{R}$ and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad (t,x) \mapsto t$.

2.1 Logarithmic scales

In this subsection we define and investigate logarithmic scales. A *logarithmic scale* is a tuple of functions where each component is defined by taking logarithm of the previous one and then "translating" by a definable function depending only on t starting at $C \to C, \quad (t,x) \mapsto x - \Theta_0(t)$, where Θ_0 is definable. This means that the components of a logarithmic scale $\mathcal{Y}(t,-)$ behaves as iterated logarithms independently of t.

Logarithmic scales are the main technical tools in the cellwise preparation of log-analytic functions and help enormously to describe how log-analytic functions depend on the last variable x.

Let $C \subset \mathbb{R}^{n+1}$ be definable.

Definition 2.1 *Let $r \in \mathbb{N}_0$. A tuple $\mathcal{Y} := (y_0, \ldots, y_r)$ of functions on C is called an r**–logarithmic scale** on C with **center** $\Theta = (\Theta_0, \ldots, \Theta_r)$ if the following hold:*

- *(a) $y_j > 0$ or $y_j < 0$ for every $j \in \{0, \ldots, r\}$.*
- *(b) Θ_j is a definable function on $\pi(C)$ for every $j \in \{0, \ldots, r\}$.*
- *(c) We have $y_0(t,x) = x - \Theta_0(t)$ and inductively $y_j(t,x) = \log(|y_{j-1}(t,x)|) - \Theta_j(t)$ for every $j \in \{1, \ldots, r\}$ and all $(t,x) \in C$.*
- *(d) Either there is $\epsilon_0 \in \,]0,1[$ such that $0 < |y_0(t,x)| < \epsilon_0|x|$ for all $(t,x) \in C$ or $\Theta_0 = 0$, and for every $j \in \{1, \ldots, r\}$ either there is $\epsilon_j \in \,]0,1[$ such that $0 < |y_j(t,x)| < \epsilon_j|\log(|y_{j-1}(t,x)|)|$ for all $(t,x) \in C$ or $\Theta_j = 0$.*

We also write y_0 instead of (y_0) for a 0–logarithmic scale.

Note that Definition 4.1 is a little bit more precise than the objects introduced in [9]: Lion and Rolin supposed that either $\Theta_j = 0$ or there is a positive constant K such that $|y_j(t,x)| \leq K\Theta_j(t)$ for every $(t,x) \in C$ and $j \in \{0, \ldots, n\}$. But for the rigorous proof of the results of Kaiser–Opris in [8], Opris [11] and [12] the full inequality in (d) is needed. With the objects from [9] one can for example not prove that $\Theta = 0$ in general on a simple cell C. (See [8, Proposition 2.19]; see [8, Definition 2.15] for the notion of a simple cell C.)

Remark 2.2 *Let $r \in \mathbb{N}_0$. Let $\mathcal{Y}$ be an r–logarithmic scale on C with center Θ. Then Θ is uniquely determined by $\mathcal{Y}$ and $\mathcal{Y}$ is log-analytic if and only if Θ is log-analytic.*

Definition 2.3 *Let $q = (q_0, \ldots, q_r) \in \mathbb{Q}^{r+1}$. We set*

$$|\mathcal{Y}|^{\otimes q} := \prod_{j=0}^{r} |y_j|^{q_j}$$

and, for $(t,x) \in C$,

$$|\mathcal{Y}(t,x)|^{\otimes q} := \prod_{j=0}^{r} |y_j(t,x)|^{q_j}.$$

Definition 2.4 *Let $\mathcal{D}$ be a decomposition of C into finitely many definable cells. Let $f : C \to \mathbb{R}$ be a function. We set*

$$C^f := \{(t, f(t,x)) \in \mathbb{R}^n \times \mathbb{R} \mid (t,x) \in C\}$$

and

$$\mathcal{D}^f := \{D^f \mid D \in \mathcal{D}\}.$$

Remark 2.5 *The following hold.*

(1) *Let $f : C \to \mathbb{R}$ be definable. Then $D := C^f$ is definable.*

(2) *Let $l \in \{1, \ldots, r\}$. We define $\mu_l : C \to \mathbb{R}, (t, x) \mapsto x - \Theta_l(t)$, and inductively for $j \in \{l+1, \ldots, r\}$ we define $\mu_j : C \to \mathbb{R}, (t, x) \mapsto \log(|\mu_{j-1}(t, x)|) - \Theta_j(t)$. Then $\mathcal{Y}_{r-l,D} := (\mu_l, \ldots, \mu_r)$ is a well-defined $(r-l)$–logarithmic scale with center $(\Theta_l, \ldots, \Theta_r)$ on $D := C^{\log(|y_{l-1}|)}$.*

Proof Property (1) is clear. For property (2) note that

$$\mathcal{Y}(t, x) = (y_0(t, x), \ldots, y_{l-1}(t, x), \mathcal{Y}_{r-l,D}(t, \log(|y_{l-1}(t, x)|)))$$

for every $(t, x) \in C$ where $y_0(t, x) = x - \Theta_0(x)$ and inductively

$$y_j(t, x) = \log(|y_{j-1}(t, x)|) - \Theta_j(x)$$

for $j \in \{1, \ldots, r\}$. So it is straightforward to see with Definition 2.1 that $\mathcal{Y}_{r-l,D}$ is an $(r-l)$–logarithmic scale on D. □

Definition 2.6 *Let $M \in \mathbb{R}_{>0}$. We set*

$$C_{>M}(\mathcal{Y}) := \{(t, x) \in C \mid |y_l(t, x)| > M \text{ for all } l \in \{1, \ldots, r\}\}$$

and, for every $l \in \{1, \ldots, r\}$,

$$C_{l,M}(\mathcal{Y}) := \{(t, x) \in C \mid |y_l(t, x)| \leq M\}.$$

Remark 2.7 *Let $M \in \mathbb{R}_{>0}$. Then $C_{>M}$ and $C_{1,M}, \ldots, C_{r,M}$ are definable.*

Remark 2.8 *Assume $r \in \mathbb{N}$. The following properties hold.*

(1) *Let $l \in \{1, \ldots, r\}$. We have*

$$|y_l| \leq |\log(|y_{l-1}|)|$$

on C.

(2) *Let $c, \lambda_1, \ldots, \lambda_r \in \mathbb{R}$. Then there is $M \in \mathbb{R}_{>1}$ such that*

$$|c + \sum_{k=1}^{r} \lambda_k \log(|y_k|)| \leq \frac{|y_1|}{2}$$

on $C_{>M}$.

Proof Recall that $y_0(t,x) = x - \Theta_0(t)$ and inductively for $j \in \{1,\dots,r\}$ $y_j(t,x) = \log(|y_{j-1}(t,x)|) - \Theta_j(t)$ for every $(t,x) \in C$.

(1) We find $\epsilon_l \in]0,1[$ such that $|y_l(t,x)| \le \epsilon_l |\log(|y_{l-1}(t,x)|)|$ for every $(t,x) \in C$ or $\Theta_l = 0$ by Definition 2.1. Hence $|y_l| \le |\log(|y_{l-1}|)|$.

(2) Take $M \ge \exp_r(1)$ to obtain with (1)

$$|c + \sum_{k=1}^{r} \lambda_k \log(|y_k|)| \le |c| + \sum_{k=1}^{r} |\lambda_k| \cdot \log_k(|y_1|)$$

on $C_{>M}$. By increasing M if necessary we may assume $|c| \le \frac{|y_1|}{4}$ and

$$|\lambda_l| \log_l(|y_1|) \le \frac{|y_1|}{4r}$$

for every $l \in \{1,\dots,r\}$ on $C_{>M}$. This gives the result. $\square$

The rest of Section 2.2 is important for technical proofs of the main results of this paper.

Definition 2.9 *Let $m \in \mathbb{N}$. Let $D \subset X \subset \mathbb{R}^m$. Let $f, g : X \to \mathbb{R}$ be functions. We call f* ***similar to*** *g* ***on*** *D, written $f \sim_D g$, if there is $\delta \in \mathbb{R}_{>0}$ such that $1/\delta \cdot g < f < \delta \cdot g$ on D.*

Remark 2.10 *Let $D \subset X \subset \mathbb{R}^m$. Let $f, g : X \to \mathbb{R}$ be functions. If $f \sim_D g$ then f and g don't have a zero on D.*

Remark 2.11 *Let $D \subset X \subset \mathbb{R}^m$ and let $f, g : X \to \mathbb{R}$ be functions.*

(1) *It holds $f \sim_D g$ if and only if $f/g \sim_D 1$.*
(2) *If $f \sim_D g$ then $\mathrm{sign}(f) = \mathrm{sign}(g)$ on D.*
(3) *The relation $f \sim_D g$ is an equivalence relation on the set of all functions on X without a zero on D.*
(4) *The set of functions on X which are similar to 1 on D form a divisible group with respect to pointwise multiplication.*

Remark 2.12 *If $\Theta_0 \neq 0$ then $x \sim_C \Theta_0$. Let $j \in \{1,\dots,r\}$. If $\Theta_j \neq 0$ then $\log(|y_{j-1}|) \sim_C \Theta_j$.*

Proof Assume $\Theta_0 \neq 0$. Then there is $\epsilon \in]0,1[$ such that $|x - \Theta_0(t)| < \epsilon |x|$ for every $(t,x) \in C$. This gives that $x \neq 0$ and $1 - \epsilon < \Theta_0(t)/x < 1 + \epsilon$ for every $(t,x) \in C$. Set $\delta := \max\{1/(1-\epsilon), 1+\epsilon\}$. Then $1/\delta < \Theta_0(t)/x < \delta$ for every $(t,x) \in C$.

Assume $\Theta_j \neq 0$. Then there is $\epsilon_j \in]0,1[$ such that

$$|\log(|y_{j-1}(t,x)|) - \Theta_j(t)| < \epsilon_j |\log(|y_{j-1}(t,x)|)|$$

for every $(t,x) \in C$. This gives $\log(|y_{j-1}(t,x)|) \neq 0$ for every $(t,x) \in C$. Now proceed in the same way as above. $\square$

Proposition 2.13 *Let $\Psi : \pi(C) \to \mathbb{R}$ be definable. The following properties hold.*

(1) Let $l \in \{1,\ldots,r\}$. If $|y_{l-1}| \sim_{C_{>M}} \Psi$ for some $M > 1$ then there is $N \geq M$ such that $|y_l| \sim_{C_{>N}} |\log(\Psi) - \Theta_l|$.

(2) Let E be a set of positive definable functions on $\pi(C)$ such that Ψ and $\Theta_1,\ldots,\Theta_r$ can be constructed from E. Let $q_1,\ldots,q_r \in \mathbb{Q}$.

(i) Let $|y_1| \sim_{C_{>M}} \Psi$ for some $M > 1$. There is $N \geq M$ and a function $\mu : \pi(C) \to \mathbb{R}_{>0}$ which can be constructed from E such that

$$\prod_{j=1}^{r} |y_j|^{q_j} \sim_{C_{>N}} \mu.$$

(ii) Suppose

$$y_0 \sim_C \Psi \prod_{j=1}^{r} |y_j|^{q_j}.$$

Then there is $M \in \mathbb{R}_{>1}$ such that $|y_1| \sim_{C_{>M}} |\log(|\Psi|) - \Theta_1|$. Additionally there is $N \geq M$ and a function $\xi : \pi(C) \to \mathbb{R}$ which can be constructed from E such that $y_0 \sim_{C_{>N}} \xi$.

Proof (1) Let $\delta > 0$ be with $|y_{l-1}|/\delta < \Psi < \delta|y_{l-1}|$ on $C_{>M}$. Note that $\delta > 1$. By taking logarithm and subtracting Θ_l we get

$$-\log(\delta) + y_l < \log(\Psi) - \Theta_l < \log(\delta) + y_l$$

on $C_{>M}$. Set $N := \max\{M, 2\log(\delta)\}$. If $y_l > 0$ on C we obtain on $C_{>N}$ that $-\log(\delta) + y_l > 0$ and therefore

$$|y_l| - \log(\delta) < |\log(\Psi) - \Theta_l| < |y_l| + \log(\delta).$$

If $y_l < 0$ on C we obtain on $C_{>N}$ that $\log(\delta) + y_l < 0$ and therefore

$$|y_l| - \log(\delta) \leq |\log(\delta) + y_l| < |\log(\Psi) - \Theta_l| < |-\log(\delta) + y_l| \leq |y_l| + \log(\delta).$$

In both cases we obtain

$$\frac{|y_l|}{2} < |\log(\Psi) - \Theta_l| < 2|y_l|$$

on $C_{>N}$.

(2i) Let $\Psi_1 := \Psi$. With (1) we find inductively for $l \in \{2, \ldots, r\}$ a real number $N_l \geq N_{l-1}$ such that

$$|y_l| \sim_{C_{>N_l}} |\log(\Psi_{l-1}) - \Theta_l| := \Psi_l$$

where $N_1 := M$. We see with an easy induction on $l \in \{1, \ldots, r\}$ and Remark 1.8 that Ψ_l can be constructed from E for every $l \in \{1, \ldots, r\}$. For $N := N_r$ we obtain with (4) in Remark 2.11

$$|y_1|^{q_1} \cdot \ldots \cdot |y_r|^{q_r} \sim_{C_{>N}} \Psi_1^{q_1} \cdot \ldots \cdot \Psi_r^{q_r} := \mu.$$

Note that $\mu : \pi(C) \to \mathbb{R}_{>0}$ can be constructed from E by Remark 1.8.

(2ii) Let $\delta > 0$ be such that

$$\frac{1}{\delta}|y_1|^{q_1} \cdot \ldots \cdot |y_r|^{q_r}\Psi < y_0 < \delta|y_1|^{q_1} \cdot \ldots \cdot |y_r|^{q_r}\Psi$$

on C. Let $\kappa := \max\{1/\delta, \delta\}$. Taking absolute values, logarithm and subtracting Θ_1 we obtain $-\log(\kappa) + L + \log(|\Psi|) - \Theta_1 < y_1 < \log(\kappa) + L + \log(|\Psi|) - \Theta_1$ on C where $L := \sum_{k=1}^r q_k \log(|y_k|)$. By Remark 2.8 we find $M > 1$ such that

$$\log(\kappa) + |L| \leq \frac{|y_1|}{2}$$

on $C_{>M}$. We obtain

$$-\frac{|y_1|}{2} + \log(|\Psi|) - \Theta_1 < y_1 < \frac{|y_1|}{2} + \log(|\Psi|) - \Theta_1$$

and therefore

$$\frac{|y_1|}{2} < |\log(|\Psi|) - \Theta_1| < 2|y_1|$$

on $C_{>M}$. Let $\Gamma := |\log(|\Psi|) - \Theta_1|$. Then $\Gamma : \pi(C) \to \mathbb{R}_{>0}$ can be constructed from E, and it holds $|y_1| \sim_C \Gamma$. By $(2), (i)$ we find a function $\mu : \pi(C) \to \mathbb{R}_{>0}$ which can be constructed from E and $N \geq M$ such that $|y_1|^{q_1} \cdot \ldots \cdot |y_r|^{q_r} \sim_{C_{>N}} \mu$. Therefore with (1) in Remark 2.11 $\Psi \cdot |y_1|^{q_1} \cdot \ldots \cdot |y_r|^{q_r} \sim_{C_{>N}} \Psi \cdot \mu$ and by (3) in Remark 2.11 $y_0 \sim_{C_{>N}} \Psi \cdot \mu$. So take $\xi := \Psi \cdot \mu$. Then ξ can be constructed from E. □

2.2 A preparation theorem for log-analytic functions

In this section we present the results from Lion–Rolin [9], ie preparation theorems for globally subanalytic functions and log-analytic functions.

Definition 2.14 (Lion–Rolin [9, Section 0.4]) *Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be globally subanalytic. Let $f : C \to \mathbb{R}$ be a function. Then f is called **globally subanalytically prepared in x with center θ** if for every $(t,x) \in C$*

$$f(t,x) = a(t) \cdot |x - \theta(t)|^q \cdot u(t,x)$$

where $q \in \mathbb{Q}$, $\theta : \pi(C) \to \mathbb{R}$ is a globally subanalytic function such that either $x > \theta(t)$ for every $(t,x) \in C$ or $x < \theta(t)$ for every $(t,x) \in C$, a is a globally subanalytic function on $\pi(C)$ which is identically zero or has no zeros, and the following holds for u. There is $s \in \mathbb{N}$ such that $u = v \circ \phi$, where v is a power series which converges on an open neighborhood of $[-1,1]^s$ with $v([-1,1]^s) \subset \mathbb{R}_{>0}$, and $\phi := (\phi_1, \ldots, \phi_s) : C \to [-1,1]^s$ with $\phi_j(t,x) = b_j(t)|x - \theta(t)|^{p_j}$ for $(t,x) \in C$, where $p_j \in \mathbb{Q}$ and $b_j : \pi(C) \to \mathbb{R}$ is globally subanalytic for $j \in \{1, \ldots, s\}$. Additionally, either there is $\epsilon \in \,]0,1[$ such that $|x - \theta(t)| < \epsilon|x|$ for every $(t,x) \in C$, or $\theta = 0$.

We see that a globally subanalytic prepared function has roughly speaking the form of a Puiseux series in one variable.

Fact 2.15 (Lion–Rolin [9, Section 1]) *Let $m \in \mathbb{N}$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ and $f_1, \ldots, f_m : X \to \mathbb{R}$ be globally subanalytic. Then there is a globally subanalytic cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is a globally subanalytic $\theta : \pi(C) \to \mathbb{R}$ such that $f_1, \ldots, f_m$ are globally subanalytically prepared in x with center θ on C.*

Remark 2.16 *There are similar versions of this preparation theorem for reducts of $\mathbb{R}_{\text{an}}$ like $\mathbb{R}_{\mathcal{W}}$ where $\mathcal{W}$ is a convergent Weierstrass system. (Compare with Miller [10] for the details).*

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be a definable cell.

Definition 2.17 (Lion–Rolin [9, Section 0.4]) *Let $r \in \mathbb{N}_0$. Let $g : C \to \mathbb{R}$ be a function. We say that g is **r–log-analytically prepared in x with center Θ** if*

$$g(t,x) = a(t)|\mathcal{Y}(t,x)|^{\otimes q} u(t,x)$$

for all $(t,x) \in C$ where a is a definable function on $\pi(C)$ which vanishes identically or has no zero, $\mathcal{Y} = (y_0, \ldots, y_r)$ is an r–logarithmic scale with center Θ on C, $q \in \mathbb{Q}^{r+1}$ and the following holds for u. There is $s \in \mathbb{N}$ such that $u = v \circ \phi$, where v is a power series which converges on an open neighborhood of $[-1,1]^s$ with $v([-1,1]^s) \subset \mathbb{R}_{>0}$, and $\phi := (\phi_1, \ldots, \phi_s) : C \to [-1,1]^s$ is a function of the form

$$\phi_j(t,x) := b_j(t)|\mathcal{Y}(t,x)|^{\otimes p_j}$$

for $j \in \{1,\dots,s\}$ and $(t,x) \in C$, where $b_j : \pi(C) \to \mathbb{R}$ is definable and $p_j := (p_{j0},\dots,p_{jr}) \in \mathbb{Q}^{r+1}$. We call a ***coefficient*** *and $b := (b_1,\dots,b_s)$ a tuple of* ***base functions*** *for f. An* ***LA-preparing tuple*** *for f is then*

$$\mathcal{J} := (r,\mathcal{Y},a,q,s,v,b,P)$$

where
$$P := \begin{pmatrix} p_{10} & \cdot & \cdot & p_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{s0} & \cdot & \cdot & p_{sr} \end{pmatrix} \in M\big(s \times (r+1), \mathbb{Q}\big).$$

Remark 2.18 *Let $f : C \to \mathbb{R}$ be a function. Let $f \sim_C g$ where g is r–log-analytically prepared in x with preparing tuple $(r,\mathcal{Y},a,q,s,v,b,P)$. Then*

$$f \sim_C a|\mathcal{Y}|^{\otimes q}.$$

Definition 2.19 *Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let $g_1,\dots,g_m : C \to \mathbb{R}$ be functions. We say that $g_1,\dots,g_m$ are r–log-analytically prepared in x* ***in a simultaneous way*** *if there is $\Theta : \pi(C) \to \mathbb{R}^{r+1}$ such that $g_1,\dots,g_m$ are r–log-analytically prepared in x with center Θ.*

Remark 2.20 *Let $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Let $f_1,\dots,f_m : C \to \mathbb{R}$ be r–log-analytically prepared in x in a simultaneous way. Then there are preparing tuples for $f_1,\dots,f_m$ which coincide in $r,\mathcal{Y},s,b,P$.*

Proof This follows immediately with Definition 2.17 and by redefining the corresponding power series $v_1,\dots,v_m$. □

Fact 2.21 (Lion–Rolin [9, Section 2.1]) *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let $f_1,\dots,f_m : X \to \mathbb{R}$ be log-analytic functions of order at most r. Then there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that $f_1|_C,\dots,f_m|_C$ are r–log-analytically prepared in x in a simultaneous way for every $C \in \mathcal{C}$.*

The rest of this section is about the following question: Are the coefficient, center and base functions of the cellwise preparation of a log-analytic function again log-analytic? We will see that this question generally has a negative answer.

Definition 2.22 *Let $\mathcal{Y}$ be an r–logarithmic scale on C with center $\Theta = (\Theta_0,\dots,\Theta_r)$. We call $\mathcal{Y}$* ***pure*** *if its center Θ is log-analytic.*

Definition 2.23 *Let $r \in \mathbb{N}_0$.*

(a) *Let C be a cell and $f : C \to \mathbb{R}$ be a function. We call $f : C \to \mathbb{R}$* ***purely r–log-analytically prepared in x with center Θ*** *if f is r–log-analytically prepared in x with log-analytic center Θ, log-analytic coefficient and log-analytic base functions. An LA-preparing tuple for g with log-analytic components is then called a* ***pure LA-preparing tuple*** *for g.*

(b) *Let $f_1, \ldots, f_m : C \to \mathbb{R}$ be functions. We call $f_1, \ldots, f_m$ purely r–log-analytically prepared in x* ***in a simultaneous way*** *if $f_1, \ldots, f_m$ are purely r–log-analytically prepared in x with the same center.*

Remark 2.24 *Let $r \in \mathbb{N}_0$. If g is r–log-analytically prepared in x then g is definable but not necessarily log-analytic. If g is purely r–log-analytically prepared in x then g is log-analytic.*

Purely r–log-analytically prepared functions in x induce the following definition of *purely log-analytic functions in x*.

Definition 2.25 *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let $f : X \to \mathbb{R}$ be a function in x.*

(a) *Let $r \in \mathbb{N}_0$. By induction on r we define that f is* ***purely log-analytic in x of order at most r.***

Base case. *The function f is purely log-analytic in x of order at most 0 if the following holds: There is a definable cell decomposition $\mathcal{C}$ of X such that for every $C \in \mathcal{C}$ there is $m \in \mathbb{N}_0$, a globally subanalytic $F : \mathbb{R}^{m+1} \to \mathbb{R}$ and a log-analytic $g : \pi(C) \to \mathbb{R}^m$ such that $f(t,x) = F(g(t),x)$ for $(t,x) \in C$.*

Inductive step. *The function f is purely log-analytic in x of order at most r if the following holds: There is a definable cell decomposition $\mathcal{C}$ of X such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, purely log-analytic functions $g_1, \ldots, g_k : C \to \mathbb{R}, h_1, \ldots, h_l : C \to \mathbb{R}_{>0}$ in x of order at most $r-1$, and a globally subanalytic function $F : \mathbb{R}^{k+l} \to \mathbb{R}$ such that*

$$f|_C = F(g_1, \ldots, g_k, \log(h_1), \ldots, \log(h_l)).$$

(b) *Let $r \in \mathbb{N}_0$. The function f is* ***purely log-analytic in x of order r*** *if f is purely log-analytic in x of order at most r but not purely log-analytic in x of order at most $r-1$.*

(c) *The function f is* ***purely log-analytic in x*** *if there is $r \in \mathbb{N}_0$ such that f is purely log-analytic in x of order r.*

Remark 2.26 *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. A purely log-analytic function in x on X is log-analytic.*

Remark 2.27 *Let $f : C \to \mathbb{R}$ be a function. Let $r \in \mathbb{N}_0$.*

(1) A log-analytic function $g : C \to \mathbb{R}$ of order at most r is purely log-analytic in x of order at most r.

(2) A purely r–log-analytically prepared function $g : C \to \mathbb{R}$ in x is purely log-analytic in x of order at most r.

Now we can give the promised example that the above Fact 2.21 can in general not be carried out in the log-analytic category.

Example 2.28 (Kaiser–Opris, [8]) *Let $\phi\colon\ ]0,\infty[\to \mathbb{R},\quad y \mapsto y/(1+y)$. Consider the log-analytic function*

$$f : \mathbb{R}_{>0} \times \mathbb{R}, \quad (t,x) \mapsto -\frac{1}{\log(\phi(x))} - t.$$

Then f is log-analytic of order 1, but does not allow piecewise a pure 1–log-analytic preparation in x.

Proof For the reader's convenience we present the proof from [8] here. Assume that the contrary holds. Let $\mathcal{C}$ be the corresponding cell decomposition. Let

$$\psi\colon\ ]0,1[\to \mathbb{R}, \quad y \mapsto y/(1-y).$$

Then ψ is the compositional inverse of ϕ. Note that $f(t,\psi(e^{-1/t})) = 0$ for all $t \in \mathbb{R}_{>0}$. Let $\alpha : \mathbb{R}_{>0} \to \mathbb{R}, t \mapsto \psi(e^{-1/t})$. Then α is not log-analytic and $\alpha(t) = \sum_{n=1}^{\infty} e^{-n/t}$ for all $t \in \mathbb{R}_{>0}$. By passing to a finer definable cell decomposition we find a cell C of the form $C := \{(t,x) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid 0 < t < \epsilon, \alpha(t) < x < \alpha(t) + \eta(t)\}$ with some suitable $\epsilon \in \mathbb{R}_{>0}$ and some definable function $\eta :]0,\epsilon[\to \mathbb{R}_{>0}$ such that f is purely 1–log-analytically prepared in x on C. Let $(1, \mathcal{Y}, a, q, s, v, b, P)$ be a pure preparing tuple for $f|_C$ and let $\Theta = (\Theta_0, \Theta_1)$ be the center of $\mathcal{Y}$.

Claim 2.29 $\Theta_0 = 0$.

Proof Assume that Θ_0 is not the zero function. By the definition of a 1–logarithmic scale we find $\varepsilon_0 \in]0,1[$ such that $|y_0| < \varepsilon_0|x|$ on C. This implies $|\alpha(t) - \Theta_0(t)| \le \epsilon_0\alpha(t)$ for all $0 < t < \epsilon$. But this is not possible since we have $\lim_{t\searrow 0} \alpha(t)/\Theta_0(t) = 0$ by the assumption that Θ_0 is log-analytic and not the zero function. □

From

$$f(t,x) = a(t)|\mathcal{Y}(t,x)|^{\otimes q} u(t,x)$$

for all $(t,x) \in C$ (where $u : C \to \mathbb{R}_{>0}$ is suitable with $u \sim_C 1$), and

$$\lim_{x \searrow \alpha(t)} f(t,x) = 0$$

for all $t \in]0,\epsilon[$ we get by o–minimality that there is, after shrinking $\epsilon > 0$ if necessary, some $j \in \{0,1\}$ such that $\lim_{x \searrow \alpha(t)} |y_j(t,x)|^{q_j} = 0$ for all $t \in]0,\varepsilon[$. By Claim 2.29 the case $j = 0$ is not possible. In the case $j = 1$ we have, again by Claim 2.29, that $q_1 > 0$ and therefore $\Theta_1 = \log(\alpha)$. But this is a contradiction to the assumption that the function Θ_1 is log-analytic, since the function $\log(\alpha)$ on the right is not log-analytic. This can be seen by applying the logarithmic series. We obtain $\lim_{t \searrow 0}(\log(\alpha(t)) + 1/t)/e^{-1/t} \in \mathbb{R} \setminus \{0\}$. □

Let f, C, α and ψ be as in Example 2.28. In the following we mention that f can be 0–log-analytically prepared on C (after shrinking ϵ and η if necessary) with coefficient, center and base functions which can be constructed from a set E of positive definable functions with the following property: Every $g \in \log(E)$ is a component of the center of a logarithmic scale on C.

For $(w_2, w_3, w_4) \in \mathbb{R}^3$ with $w_2, w_4 > 0$, $w_4/w_2 \in]1/2, 3/2[$ and $1/(1+w_4) \in]1/2, 3/2[$ we set

$$u(w_2,w_3,w_4) := \log^*(\frac{w_4}{w_2}) + \log^*(\frac{1}{1+w_4}) + w_3$$

where $$\log^* : \mathbb{R} \to \mathbb{R}, y \mapsto \begin{cases} \log(y), & y \in [1/2, 3/2], \\ 0, & \text{otherwise.} \end{cases}$$

Consider:

$$F : \mathbb{R}^4 \to \mathbb{R},$$

$$(w_1,\ldots,w_4) \mapsto \begin{cases} -\frac{1}{u(w_2,w_3,w_4)} - w_1, & w_2, w_4 > 0, \quad w_4/w_2 \in]1/2,3/2[\,, \\ & 1/(1+w_4) \in]1/2,3/2[\,, \\ & \text{and } u(w_2,w_3,w_4) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Note that F is globally subanalytic since $\log^*$ is globally subanalytic. By shrinking ϵ and η if necessary we obtain $x/\alpha(t) \in]1/2, 3/2[$, $1/(1+x) \in]1/2, 3/2[$ and therefore

$$f(t,x) = F(t, \alpha(t), \log(\alpha(t)), x)$$

for $(t,x) \in C$. Note that α can be constructed from $E := \{\pi(C) \to \mathbb{R}_{>0}, t \mapsto e^{-1/t}\}$ (since $\alpha(t) = \psi(e^{-1/t})$ for $t \in \pi(C)$ and ψ is globally subanalytic). Therefore $\log(\alpha)$ can also be constructed from E (since $\pi(C) \to \mathbb{R}, t \mapsto \log(\psi(t))$, is log-analytic). With Fact 2.15 we find a globally subanalytic cell decomposition $\mathcal{D}$ of $\mathbb{R}^4$ such that for every $D \in \mathcal{D}$ we have that $F|_D$ is globally subanalytically prepared in x_4. By shrinking ϵ and η if necessary we may assume that there is $D \in \mathcal{D}$ such that $(t, \alpha(t), \log(\alpha(t)), x) \in D$ for every $(t,x) \in C$. So we may assume that F is globally subanalytically prepared in w_4. Consequently f is 0–log-analytically prepared in x with coefficient, center and base functions which can be constructed from E. By further shrinking ϵ and η if necessary we may assume that $|\log(x) + 1/t| < 1/2|\log(x)|$ for $(t,x) \in C$. Let $y_0 := x$ and $y_1 := \log(y_0) + 1/t$. We have that (y_0, y_1) is a 1–logarithmic scale on C with center (Θ_0, Θ_1) where $\Theta_0 := 0$ and $\Theta_1 : \pi(C) \to \mathbb{R}, t \mapsto -1/t$. So E is a set of one positive definable function on $\pi(C)$ whose logarithm coincides with Θ_1.

We will see in Section 2.3 that functions which can be constructed from a set of positive definable functions whose logarithm coincide with a component of the center of a logarithmic scale play a crucial role in preparing log-analytic functions.

2.3 A type of definable functions closed under log-analytic preparation

The main goal for this section is to prove Theorem 0.2. At first we introduce the notion of C–heirs and C–nice functions for a definable cell $C \subset \mathbb{R}^n \times \mathbb{R}$, give elementary properties and examples. A *C–heir* is a positive definable function on $\pi(C)$ whose logarithm coincides with the component of a center of a logarithmic scale on C. A *C–nice* function is a definable function which is a composition of log-analytic functions and C–heirs (ie can be constructed from a set E of C–heirs, see (c) in Definition 1.5). As Theorem 0.2 suggests, C–nice functions are crucial to understand log-analytic functions from the viewpoint of their cellwise preparation.

For Section 2.3 we need the following additional notation. "Log-analytically prepared" means always "log-analytically prepared in x". For $k, l \in \mathbb{N}$ and $s \in \{1, \ldots, l\}$ denote by $M_s(k \times l, \mathbb{Q})$ the set of all $k \times l$-matrices with rational entries where the first s columns are zero.

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable.

Definition 2.30 *We call $g : \pi(C) \to \mathbb{R}$ C**–heir** if there is $r \in \mathbb{N}_0$, an r–logarithmic scale $\mathcal{Y}$ with center $(\Theta_0, \ldots, \Theta_r)$ on C, and $l \in \{1, \ldots, r\}$ such that $g = \exp(\Theta_l)$.*

Remark 2.31 *Let $D \subset \mathbb{R}^n \times \mathbb{R}$ be definable with $\pi(C) = \pi(D)$. A C–heir is not necessarily a D–heir.*

Proof Let $C := \mathbb{R}^n \times]0, 1[$ and $D := \mathbb{R}^n \times \mathbb{R}_{\neq 0}$. Note that there is no r–logarithmic scale on D for every $r \in \mathbb{N}_0$. So a D–heir does not exist. But it is straightforward to see that (y_0, y_1) with $y_0 := x$ and $y_1 := \log(x)$ is a 1–logarithmic scale with center 0 on C and therefore that $g : \pi(C) \to \mathbb{R}, x \mapsto 1$, is a C–heir. $\square$

Definition 2.32 *We call $g : \pi(C) \to \mathbb{R}$ C–**nice** if there is a set E of C–heirs such that g can be constructed from E.*

Example 2.33 *The following hold:*

(1) A log-analytic function $f : \pi(C) \to \mathbb{R}$ is C–nice.
(2) Let $C :=]0, 1[^2$ and let $h :]0, 1[\to \mathbb{R}, t \mapsto e^{-1/t}$. Then h is not C–nice.

Proof (1) Note that f can be constructed from $E = \emptyset$. Therefore f is C–nice by Definition 2.32.

(2) Note that $\pi(C) =]0, 1[$. Suppose that h is C–nice. Let E be a set of C–heirs such that h can be constructed from E. An easy calculation shows that the center of every logarithmic scale on C vanishes (compare with Definition 2.1(d) or Proposition 2.19 in [8]). So we see $E = \emptyset$ or $E = 1$.

Claim 2.34 *The function h is log-analytic.*

Proof Suppose $E = \{1\}$. Let $e \in \mathbb{N}_0$ be such hat h has exponential number at most e with respect to E. We do an induction on e. For $e = 0$ this is clear with part (a) in Definition 1.5. So suppose $e > 0$. Let $u_1, \ldots, u_k, v_1, \ldots, v_l : \pi(C) \to \mathbb{R}$ be functions with $\exp(v_1), \ldots, \exp(v_l) \in E$ which have exponential number at most $e - 1$ with respect to E and $F : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$ be log-analytic such that

$$h(t) = F(u_1(t), \ldots, u_k(t), \exp(v_1(t)), \ldots, \exp(v_l(t)))$$

for $t \in \pi(C)$. Note that $u_1, \ldots, u_k$ are log-analytic by the inductive hypothesis and that $\exp(v_1) = \cdots = \exp(v_l) = 1$. So we obtain that h is log-analytic. If $E = \emptyset$ one sees immediately with part (a) in Definition 1.5 that h is log-analytic as a consequence that that h is C–nice. $\square$

But we have $\lim_{t \to 0} h(t)/e^{-1/t} \in \mathbb{R} \setminus \{0\}$, a contradiction to Claim 2.34 since every log-analytic function is polynomially bounded. $\square$

A C–nice function which is not log-analytic can be found in Example 2.39.

Remark 2.35 *Let $r \in \mathbb{N}_0$. Let $(\Theta_0, \ldots, \Theta_r)$ be a C–nice center of an r–logarithmic scale $\mathcal{Y}$ on C. Let $f = \exp(\Theta_j)$ for $j \in \{1, \ldots, r\}$. Then f is a C–nice C–heir.*

Proof Let E be a set of C–heirs such that $\Theta_1, \ldots, \Theta_r$ can be constructed from E. Then $\exp(\Theta_j)$ can be constructed from $E \cup \{\exp(\Theta_j)\}$ for $j \in \{1, \ldots, r\}$ by Remark 1.8 and is therefore C–nice, because $\pi(C) \mapsto \mathbb{R}, t \mapsto \exp(\Theta_j(t))$, is a C–heir. $\square$

Remark 2.36 *Let $D \subset C$ be definable.*

(1) Let $g : \pi(C) \to \mathbb{R}$ be a C–heir. Then $g|_{\pi(D)}$ is a D–heir.
(2) Let $h : \pi(C) \to \mathbb{R}$ be C–nice. Then $h|_{\pi(D)}$ is D–nice.

Proof (1) This follows from the following fact. Let $r \in \mathbb{N}_0$. Let $\mathcal{Y}$ be an r–logarithmic scale on C with center $(\Theta_0, \ldots, \Theta_r)$. Then $\mathcal{Y}|_D$ is an r–logarithmic scale with center $(\Theta_0|_{\pi(D)}, \ldots, \Theta_r|_{\pi(D)})$ on D.

(2) Let E be a set of C–heirs such that h can be constructed from E. Then it is easily seen that $h|_{\pi(D)}$ can be constructed from $E|_{\pi(D)}$ which is a set of $\pi(D)$–heirs by (1). $\square$

Remark 2.37 *The set of C–nice functions is closed under composition with log-analytic functions, ie let $m \in \mathbb{N}$ and $F : \mathbb{R}^m \to \mathbb{R}$ be log-analytic and $\eta := (\eta_1, \ldots, \eta_m) : \pi(C) \to \mathbb{R}^m$ be C–nice. Then $F \circ \eta : \pi(C) \to \mathbb{R}$ is C–nice.*

Proof Let E be a set of C–heirs such that $\eta_1, \ldots, \eta_m$ can be constructed from E. By Remark 1.8 $F(\eta_1, \ldots, \eta_m)$ can be constructed from E. $\square$

Definition 2.38 *Let $r \in \mathbb{N}_0$. Let $\mathcal{Y}$ be an r–logarithmic scale on C. Then $\mathcal{Y}$ is* ***nice*** *if its center $\Theta := (\Theta_0, \ldots, \Theta_r)$ is C–nice.*

The next example shows that not every nice logarithmic scale is log-analytic in general.

Example 2.39 *Consider the definable cell*

$$C := \{(t,x) \in \mathbb{R}^2 \mid 0 < t < 1, \tfrac{1}{1+t} + e^{-2/t + 2e^{-1/t}} < x < \tfrac{1}{1+t} + e^{-1/t}\}.$$

Then there is a nice logarithmic scale on C which is not log-analytic.

Proof We have $\pi(C) =]0,1[$, because

$$e^{-2/t+2e^{-1/t}} < e^{-1/t}$$

for every $t \in]0,1[$. Consider

$$\begin{aligned} &\Theta_0 : \pi(C) \to \mathbb{R}, t \mapsto \tfrac{1}{1+t}, \\ &\Theta_1 : \pi(C) \to \mathbb{R}, t \mapsto -1/t, \\ \text{and} \qquad &\hat{\Theta}_1 : \pi(C) \to \mathbb{R}, t \mapsto -1/t + e^{-1/t}. \end{aligned}$$

For $(t,x) \in C$ consider

$$\begin{aligned} &y_0(t,x) := x - \Theta_0(t), \\ &y_1(t,x) := \log(x - \Theta_0(t)) - \Theta_1(t), \\ \text{and} \qquad &\hat{y}_1(t,x) := \log(x - \Theta_0(t)) - \hat{\Theta}_1(t). \end{aligned}$$

Claim 2.40 *$\mathcal{Y} := (y_0, y_1)$ and $\hat{\mathcal{Y}} := (y_0, \hat{y}_1)$ are 1–logarithmic scales on C.*

Proof We have $y_0 > 0$, $y_1 < 0$, and $\hat{y}_1 < 0$ on C. Let $\epsilon_0 := 1/2$ and $\epsilon_1 := 1/2$. Let $(t,x) \in C$. We have $x < \frac{2}{1+t}$, because $e^{-1/t} < \frac{1}{1+t}$. Therefore $|x - \Theta_0(t)| < \epsilon_0 |x|$. Note that

$$e^{-2/t+2e^{-1/t}} + \frac{1}{1+t} < x.$$

So an easy calculation shows that

$$\begin{aligned} &|\log(x - \Theta_0(t)) - \Theta_1(t)| < \epsilon_1 \cdot |\log(x - \Theta_0(t))| \\ \text{and} \qquad &|\log(x - \Theta_0(t)) - \hat{\Theta}_1(t)| < \epsilon_1 \cdot |\log(x - \Theta_0(t))|. \end{aligned}$$ □

Note that Θ_0 and Θ_1 are log-analytic. Therefore $\mathcal{Y}$ is nice by (1) in Example 2.33. Note also that $\hat{\Theta}_1$ is not log-analytic. We have $\hat{\Theta}_1 = G(1/t, e^{-1/t})$ where $G : \mathbb{R}^2 \to \mathbb{R}, (w_1, w_2) \mapsto -w_1 + w_2$, is log-analytic. So $\hat{\Theta}_1$ has exponential number 1 with respect to $E := \{g\}$ where $g : \pi(C) \to \mathbb{R}, t \mapsto e^{-1/t}$ (since $t \mapsto -1/t$ is globally subanalytic). So $\hat{\Theta}_1$ can be constructed from E. By Claim 2.40 we see that g is a C–heir. Therefore $\hat{\mathcal{Y}}$ is an example for a nice 1–logarithmic scale which is not log-analytic. □

Remark 2.41 *Let $r \in \mathbb{N}_0$. Let $\mathcal{Y} := (y_0, \ldots, y_r)$ be a C–nice r–logarithmic scale with center $(\Theta_0, \ldots, \Theta_r)$ on C. Let $\Psi : \pi(C) \to \mathbb{R}$ be C–nice. Assume that*

$$y_0 \sim_C \Psi \prod_{j=1}^{r} |y_j|^{q_j}$$

where $q_1, \ldots, q_r \in \mathbb{Q}$. Then there is $M \in \mathbb{R}_{>1}$ and a C–nice $\xi : \pi(C) \to \mathbb{R}$ such that $y_0 \sim_{C_{>M}} \xi$.

Proof Let E be a set of C–heirs such that $\Psi, \Theta_0, \ldots, \Theta_r$ can be constructed from E. The assertion follows from (2),(ii) in Proposition 2.13. $\square$

Now we are able to introduce the notion of nice log-analytic prepared functions.

Definition 2.42 *Let $f : C \to \mathbb{R}$ be a function. We say that f is **nicely r–log-analytically prepared with center** Θ if f is r–log-analytically prepared with a nice r–logarithmic scale $\mathcal{Y}$ with center Θ, C–nice coefficient and C–nice base functions. A corresponding LA-preparing tuple for f is then called a **nice LA-preparing tuple** for f.*

Remark 2.43 *Let $r, m \in \mathbb{N}$ and $k \in \{1, \ldots, r\}$. Let $\mathcal{Y}_{k-1} := (y_0, \ldots, y_{k-1})$ be a nice $(k-1)$–logarithmic scale with center $(\Theta_0, \ldots, \Theta_{k-1})$ on C. Let $B := C^{\log(|y_{k-1}|)}$. Let $\alpha_1, \ldots, \alpha_m : B \to \mathbb{R}$ be nicely $(r-k)$–log-analytically prepared with center $(\Theta_k, \ldots, \Theta_r)$. For every $j \in \{1, \ldots, m\}$ consider*

$$\beta_j : C \to \mathbb{R}, (t,x) \mapsto \alpha_j(t, \log(|y_{k-1}(t,x)|)).$$

Let $\Theta := (\Theta_0, \ldots, \Theta_r)$. Then $\beta_1, \ldots, \beta_m$ are nicely r–log-analytically prepared with center Θ. Additionally there is a nice LA-preparing tuple $(r, \mathcal{Y}_r, a_j, q_j, s, v_j, b, P)$ for β_j such that $q_j \in \{0\}^k \times \mathbb{Q}^{r+1-k}$ and $P \in M_k(s \times (r+1), \mathbb{Q})$ where $\mathcal{Y}_r$ is the r–logarithmic scale on C with center $\Theta := (\Theta_0, \ldots, \Theta_r)$ for $j \in \{1, \ldots, m\}$.

Proof Let $(r-k, \hat{\mathcal{Y}}_{r-k,B}, a_j, q_j, s, v_j, b, P)$ be a nice LA-preparing tuple for α_j where $j \in \{1, \ldots, m\}$. (By Remark 2.20 s, b, P are independent from j.) Here $\hat{\mathcal{Y}}_{r-k,B}$ denotes the $(r-k)$–logarithmic scale with center $(\Theta_k, \ldots, \Theta_r)$ on B. We set

$$\mathcal{Y}_r(t,x) := (y_0(t,x), \ldots, y_{k-1}(t,x), \hat{\mathcal{Y}}_{r-k,B}(t, \log(|y_{k-1}(t,x)|)))$$

for $(t,x) \in C$. With Definition 2.1 one sees immediately that $\mathcal{Y}_r$ defines an r–logarithmic scale with center $\Theta := (\Theta_0, \ldots, \Theta_r)$ on C. Because Θ_j is C–nice for every $j \in \{0, \ldots, r\}$ we see that $\mathcal{Y}_r$ is nice. In particular it is

$$|\hat{\mathcal{Y}}_{r-k,B}(t, \log(|y_{k-1}(t,x)|))|^{\otimes q} = |\mathcal{Y}_r(t,x)|^{\otimes q^*}$$

for every $q \in \mathbb{Q}^{r-k+1}$ and $(t,x) \in C$ where $q^* := (0, \ldots, 0, q) \in \mathbb{Q}^{r+1}$. This observation gives the desired nice LA-preparing tuple for β_j where $j \in \{1, \ldots, m\}$. $\square$

Remark 2.44 *One may replace "nice" by "pure" and "nicely" by "purely" in Remark 2.43.*

Definition 2.45 *Let $r \in \mathbb{N}_0$. Let $m \in \mathbb{N}$. Let $g_1, \ldots, g_m : C \to \mathbb{R}$ be functions. We call $g_1, \ldots, g_m$ nicely r–log-analytically prepared* ***in a simultaneous way*** *if there is $\Theta : \pi(C) \to \mathbb{R}^{r+1}$ such that $g_1, \ldots, g_m$ are nicely r–log-analytically prepared with center Θ.*

Proposition 2.46 *The following properties hold.*

(1) *Let $r \in \mathbb{N}_0$. Let $g : C \to \mathbb{R}$ be nicely r–log-analytically prepared. Then there is $k \in \mathbb{N}$, a C–nice function $\eta : \pi(C) \to \mathbb{R}^k$, a globally subanalytic function $G : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}$, and a nice r–logarithmic scale $\mathcal{Y}_r$ on C such that*
$$g(t,x) = G(\eta(t), \mathcal{Y}_r(t,x))$$
for all $(t,x) \in C$.

(2) *Let $r \in \mathbb{N}$. Let $h : C \to \mathbb{R}_{>0}$ be nicely $(r-1)$–log-analytically prepared. Then there is $k \in \mathbb{N}$, a C–nice function $\eta : \pi(C) \to \mathbb{R}^k$, a globally subanalytic function $H : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}$, a nice $(r-1)$–logarithmic scale $\mathcal{Y}_{r-1} := (y_0, \ldots, y_{r-1})$ on C such that*
$$\log(h(t,x)) = H(\eta(t), \mathcal{Y}_{r-1}(t,x), \log(|y_{r-1}(t,x)|))$$
for all $(t,x) \in C$.

Proof (1) Let $(r, \mathcal{Y}_r, a, q, s, v, b, P)$ be a nice LA-preparing tuple for g. Take $k := s+1$, and
$$\eta = (\eta_1, \ldots, \eta_k) : \pi(C) \to \mathbb{R}^k, \quad t \mapsto (a(t), b_1(t), \ldots, b_s(t)).$$
Then η is C–nice. Let $z := (z_0, \ldots, z_s)$ range over $\mathbb{R}^{s+1}$ and $w := (w_0, \ldots, w_r)$ range over $\mathbb{R}^{r+1}$. Set
$$\alpha_0 : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}, \quad (z,w) \mapsto z_0 \prod_{j=0}^{r} |w_j|^{q_j}.$$
For $i \in \{1, \ldots, s\}$ let
$$\alpha_i : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}, \quad (z,w) \mapsto z_i \prod_{j=0}^{r} |w_j|^{p_{ij}}.$$
Let $G : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}$ be
$$(z_0, \ldots, z_s, w_0, \ldots, w_r) \mapsto \begin{cases} \alpha_0(z,w) v(\alpha_1(z,w), \ldots, \alpha_s(z,w)), & |\alpha_i(z,w)| \leq 1 \\ & \quad \text{for all } i \in \{1, \ldots, s\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then G is globally subanalytic and for every $(t,x) \in C$ we have

$$g(t,x) = G(\eta(t), \mathcal{Y}_r(t,x)).$$

(2) Let $(r-1, \mathcal{Y}_{r-1}, a, q, s, v, b, P)$ be a nice LA-preparing tuple for h where $\mathcal{Y}_{r-1} := (y_0, \ldots, y_{r-1})$ is a nice $(r-1)$–logarithmic scale with center $(\Theta_0, \ldots, \Theta_{r-1})$ on C. Then $a > 0$ on C. Take $k := s + r + 1$, and

$$\eta = (\eta_1, \ldots, \eta_k) : \pi(C) \to \mathbb{R}^k,$$
$$t \mapsto (\log(a(t)), b_1(t), \ldots, b_s(t), \Theta_1(t), \ldots, \Theta_{r-1}(t), 0).$$

Then η is C–nice by Remark 2.37. Let $z := (z_0, \ldots, z_{s+r})$ range over $\mathbb{R}^{s+r+1}$ and $w := (w_0, \ldots, w_r)$ over $\mathbb{R}^{r+1}$.

Set $\beta : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}, \quad (z,w) \mapsto z_0 + \sum_{j=0}^{r-1} q_j(w_{j+1} + z_{s+j+1})$.

For $i \in \{1, \ldots, s\}$ let $\alpha_i : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}, \quad (z,w) \mapsto z_i \prod_{j=0}^{r-1} |w_j|^{p_{ij}}$.

Let $H : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}$ be

$$(z_0, \ldots, z_{s+r}, w_0, \ldots, w_r) \mapsto$$
$$\begin{cases} \beta(z,w) + \log(v(\alpha_1(z,w), \ldots, \alpha_s(z,w))), & |\alpha_i(z,w)| \leq 1 \\ & \quad \text{for all } i \in \{1, \ldots, s\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then H is globally subanalytic since $\log(v)$ is globally subanalytic. For every $(t,x) \in C$ we have

$$\log(h(t,x)) = H(\eta(t), \mathcal{Y}_{r-1}(t,x), \log(|y_{r-1}(t,x)|)). \qquad \square$$

Corollary 2.47 *Let $r, k, l \in \mathbb{N}$. Let $g_1, \ldots, g_k : C \to \mathbb{R}$ and $h_1, \ldots, h_l : C \to \mathbb{R}_{>0}$ be nicely $(r-1)$–log-analytically prepared in a simultaneous way. Let $F : \mathbb{R}^{k+l} \to \mathbb{R}$ be globally subanalytic. Then there are $m \in \mathbb{N}$, a C–nice $\eta : \pi(C) \to \mathbb{R}^m$, a globally subanalytic $J : \mathbb{R}^m \times \mathbb{R}^{r+1} \to \mathbb{R}$, and a nice $(r-1)$–logarithmic scale $\mathcal{Y} := (y_0, \ldots, y_{r-1})$ on C such that for all $(t,x) \in C$,*

$$F(g_1(t,x), \ldots, g_k(t,x), \log(h_1(t,x)), \ldots, \log(h_l(t,x)))$$
$$= J(\eta(t), \mathcal{Y}(t,x), \log(|y_{r-1}(t,x)|)).$$

Remark 2.48 *One may replace "nicely prepared" by "purely prepared", "C–nice" by "log-analytic" and "nice r–logarithmic scale" by "pure r–logarithmic scale" in Proposition 2.46 and Corollary 2.47.*

Here is the promised class of log-analytic functions in x which is closed under log-analytic preparation.

Definition 2.49 *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let $f : X \to \mathbb{R}$ be a function.*

(a) *Let $r \in \mathbb{N}_0$. By induction on r we define that f is **nicely log-analytic in** x **of order at most** r:*

__Base case.__ The function f is log-analytic in x of order at most 0 if the following holds: There is a definable cell decomposition $\mathcal{C}$ of X such that for every $C \in \mathcal{C}$ there is $m \in \mathbb{N}$, a globally subanalytic $F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and a C–nice $g : \pi(C) \to \mathbb{R}^m$ such that $f(t,x) = F(g(t),x)$ for every $(t,x) \in C$.

__Inductive step.__ The function f is nicely log-analytic in x of order at most r if the following holds: There is a definable cell decomposition $\mathcal{C}$ of X such that for $C \in \mathcal{C}$ there are $k,l \in \mathbb{N}_0$, nicely log-analytic functions $g_1,\ldots,g_k : C \to \mathbb{R}, h_1,\ldots,h_l : C \to \mathbb{R}_{>0}$ in x of order at most $r-1$, and a globally subanalytic $F : \mathbb{R}^{k+l} \to \mathbb{R}$ such that $f|_C = F(g_1,\ldots,g_k,\log(h_1),\ldots,\log(h_l))$.

(b) *Let $r \in \mathbb{N}_0$. The function f is **nicely log-analytic in** x **of order** r if f is nicely log-analytic in x of order at most r but not nicely log-analytic in x of order at most $r-1$.*

(c) *The function f is **nicely log-analytic in** x if there is $r \in \mathbb{N}_0$ such that f is nicely log-analytic in x of order r.*

Remark 2.50 *Let $r \in \mathbb{N}_0$. Then the following properties hold.*

(1) *Let $f : C \to \mathbb{R}$ be a function. If f is purely log-analytic in x of order at most r then it is nicely log-analytic in x of order at most r.*

(2) *Let $m \in \mathbb{N}$. Let $f_1,\ldots,f_m : C \to \mathbb{R}$ be nicely log-analytic in x of order at most r and let $F : \mathbb{R}^m \to \mathbb{R}$ be globally subanalytic. Then $F(f_1,\ldots,f_m)$ is nicely log-analytic in x of order at most r.*

(3) *Let $m \in \mathbb{N}$. Let $f_1,\ldots,f_m : C \to \mathbb{R}$ be nicely log-analytic in x and let $F : \mathbb{R}^m \to \mathbb{R}$ be log-analytic. Then $F(f_1,\ldots,f_m)$ is nicely log-analytic in x.*

Remark 2.51 *A nice logarithmic scale $\mathcal{Y} := (y_0,\ldots,y_r)$ is nicely log-analytic. In particular y_j is nicely log-analytic in x of order at most j for $j \in \{0,\ldots,r\}$.*

Proof Let $\mathcal{Y} := (y_0,\ldots,y_r)$. Let $(\Theta_0,\ldots,\Theta_r)$ be the C–nice center of $\mathcal{Y}$. We show by induction on $j \in \{0,\ldots,r\}$ that y_j is nicely log-analytic in x of order at most j.

$j = 0$: We have $y_0(t,x) = x - \Theta_0(t)$ for every $(t,x) \in C$. That y_0 is nicely log-analytic in x of order 0 is clear with (2) in Remark 2.50.

$j-1 \to j$: It holds $y_j(t,x) = \log(|y_{j-1}(t,x)|) - \Theta_j(t)$ for every $(t,x) \in C$. That y_j is nicely log-analytic in x of order at most j is clear with (2) in Remark 2.50. □

Proposition 2.52 *Let $k \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Let $F : \mathbb{R}^{k+r+1} \to \mathbb{R}$ be globally subanalytic and $\eta : \pi(C) \to \mathbb{R}^k$ be C–nice. Let $\mathcal{Y} := (y_0, \ldots, y_r)$ be a nice r–logarithmic scale with center $(\Theta_0, \ldots, \Theta_r)$ on C. Let*

$$f : C \to \mathbb{R}, (t,x) \mapsto F(\eta(t), \mathcal{Y}(t,x)).$$

Then the following properties hold.

(1) *f is nicely log-analytic in x of order at most r.*

(2) *Assume that there is a C–nice $\xi : \pi(C) \to \mathbb{R}$ such that $y_0 \sim_C \xi$. Then f is nicely log-analytic in x of order at most $r-1$.*

Proof (1) By Remark 2.51 y_l is nicely log-analytic in x of order at most l for every $l \in \{0, \ldots, r\}$. We are done with (2) in Remark 2.50.

(2) By (1) in Remark 2.11 there is $\delta > 1$ such that $1/\delta < \frac{y_0}{\xi} < \delta$ on C. Set

$$\log^* : \mathbb{R} \to \mathbb{R}, \quad y \mapsto \begin{cases} \log(y), & y \in [1/\delta, \delta], \\ 0, & y \notin [1/\delta, \delta]. \end{cases}$$

Then $\log^*$ is globally subanalytic. We have $y_1 = y_1^*$ on C where

$$y_1^* := \log^*\big(\frac{y_0}{\xi}\big) + \log(|\xi|) - \Theta_1.$$

Then y_1^* is nicely log-analytic in x of order 0, because $\log(|\xi|)$ is C–nice by Remark 2.37. In particular, $f(t,x) = F(\eta(t), y_0(t,x), y_1^*(t,x), \ldots, y_r^*(t,x))$ for every $(t,x) \in C$, where inductively for $l \in \{2, \ldots, r-1\}$, $y_l^* := \log(|y_{l-1}^*|) - \Theta_l$. Note that y_l^* is nicely log-analytic in x of order at most $l-1$ for every $l \in \{2, \ldots, r\}$ and therefore $F(\eta, y_0, y_1^*, \ldots, y_r^*)$ is nicely log-analytic in x of order at most $r-1$ by (2) in Remark 2.50. □

Remark 2.53 *Let $r \in \mathbb{N}_0$. Let $f : C \to \mathbb{R}$ be a function. If f is nicely r–log-analytically prepared in x then f is nicely log-analytic in x of order at most r.*

Proof By (1) in Proposition 2.46 there is $k \in \mathbb{N}$, a C–nice function $\eta : \pi(C) \to \mathbb{R}^k$, a globally subanalytic function $G : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}$, and a nice r–logarithmic scale $\mathcal{Y}$ on C such that $g(t,x) = G(\eta(t), \mathcal{Y}(t,x))$ for all $(t,x) \in C$. By (1) in Proposition 2.52, g is nicely log-analytic in x of order at most r. □

Proposition 2.54 *Let $k \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Let $F : \mathbb{R}^{k+r} \to \mathbb{R}$ be globally subanalytic and $\eta : \pi(C) \to \mathbb{R}^k$ be C–nice. Let $\mathcal{Y} := (y_0, \ldots, y_r)$ be a nice r–logarithmic scale with center $(\Theta_0, \ldots, \Theta_r)$ on C. Let*

$$f : C \to \mathbb{R}, \quad (t,x) \mapsto F(\eta(t), y_1(t,x), \ldots, y_r(t,x)).$$

Then the following hold.

(1) *There is a nicely log-analytic function $\kappa : C^{\log(|y_0|)} \to \mathbb{R}$ in x of order at most $r-1$ such that $f(t,x) = \kappa(t, \log(|y_0(t,x)|))$ for every $(t,x) \in C$.*

(2) *Assume $r \geq 2$. Let $l \in \{1, \ldots, r-1\}$. Let $\xi : \pi(C) \to \mathbb{R}$ be C–nice such that $y_l \sim_C \xi$. Then there is a nicely log-analytic function $\lambda : C^{\log(|y_0|)} \to \mathbb{R}$ in x of order at most $r-2$ such that $f(t,x) = \lambda(t, \log(|y_0(t,x)|))$ for every $(t,x) \in C$.*

Proof We set $B := C^{\log(|y_0|)}$. Let $\mu_0 : B \to \mathbb{R}, (t,x) \mapsto x - \Theta_1(t)$ and inductively for $j \in \{1, \ldots, r-1\}$ let $\mu_j : B \to \mathbb{R}, (t,x) \mapsto \log(|\mu_{j-1}(t,x)|) - \Theta_{j+1}(t)$. Note that $y_j(t,x) = \mu_{j-1}(t, \log(|y_0(t,x)|))$ for every $(t,x) \in C$ and $j \in \{1, \ldots, r\}$. With Remark 2.5 we obtain that $\mathcal{Y}_{r-1,B} := (\mu_0, \ldots, \mu_{r-1})$ is a nice $(r-1)$–logarithmic scale with center $(\Theta_1, \ldots, \Theta_r)$ on B. It follows that $f(t,x) = F(\eta(t), \mathcal{Y}_{r-1,B}(t, \log(|y_0(t,x)|)))$ for every $(t,x) \in C$.

(1) We set $\kappa : B \to \mathbb{R}$, $(t,x) \mapsto F(\eta(t), \mathcal{Y}_{r-1,B}(t,x))$. By (1) in Proposition 2.52 κ is a nicely log-analytic function in x of order at most $r-1$ and we obtain

$$f(t,x) = \kappa(t, \log(|y_0(t,x)|))$$

for all $(t,x) \in C$.

(2) Since $y_l \sim_C \xi$ we obtain $\mu_{l-1} \sim_B \xi$ since ξ depends only on t and $\pi(B) = \pi(C)$. Thus by (1) in Remark 2.11 there is $\delta > 1$ such that $1/\delta < \mu_{l-1}/\xi < \delta$ on B. Consider

$$\log^*\colon\ \mathbb{R} \to \mathbb{R}, \quad y \mapsto \begin{cases} \log(y), & y \in [1/\delta, \delta], \\ 0, & y \notin [1/\delta, \delta]. \end{cases}$$

Then $\log^*$ is globally subanalytic. Set

$$\mu_l^* := \log^*\Big(\frac{\mu_{l-1}}{\xi}\Big) + \log(|\xi|) - \Theta_{l+1}$$

and inductively, for $j \in \{l+1, \ldots, r-1\}$, $\mu_j^* := \log(|\mu_{j-1}^*|) - \Theta_{j+1}$. Then by Remark 2.51 and (2) in 2.50 μ_l^* is a nicely log-analytic function in x of order at most $l-1$, because $\log(|\xi|)$ is C–nice by Remark 2.37. We see similarly as in the proof of Remark 2.51 that μ_j^* is nicely log-analytic in x of order at most $j-1$ for every $j \in \{l+1, \ldots, r-1\}$. We set

$$\lambda\colon\ B \to \mathbb{R}, \quad (t,x) \mapsto F(\eta(t), \mu_0(t,x), \ldots, \mu_{l-1}(t,x), \mu_l^*(t,x), \ldots, \mu_{r-1}^*(t,x)).$$

By (2) in Remark 2.50 λ is nicely log-analytic in x of order at most $r-2$. Because $\mu_j = \mu_j^*$ for every $j \in \{l, \ldots, r-1\}$ on B we obtain $f(t,x) = \lambda(t, \log(|y_0(t,x)|))$ for every $(t,x) \in C$. □

An immediate consequence from the globally subanalytic preparation theorem is the following.

Proposition 2.55 *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $m \in \mathbb{N}$. Let $f_1, \ldots, f_m \colon X \to \mathbb{R}$ be nicely log-analytic functions in x of order 0. Then there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that $f_1|_C, \ldots, f_m|_C$ are nicely 0–log-analytically prepared in a simultaneous way for every $C \in \mathcal{C}$.*

Proof It is enough to consider the following situation: Let $g_1, \ldots, g_k \colon \pi(X) \to \mathbb{R}$ be X–nice and $F_j : \mathbb{R}^{k+1} \to \mathbb{R}$ be globally subanalytic such that

$$f_j(t,x) = F_j(g_1(t), \ldots, g_k(t), x)$$

for every $(t,x) \in X$ and $j \in \{1, \ldots, m\}$. Let $g := (g_1, \ldots, g_k)$. Let $z := (z_1, \ldots, z_k)$ range over $\mathbb{R}^k$. Let $\pi^* \colon \mathbb{R}^{k+1} \to \mathbb{R}^k, \quad (z,x) \mapsto z$. With Fact 2.15 we find a globally subanalytic cell decomposition $\mathcal{D}$ of $\mathbb{R}^k \times \mathbb{R}_{\neq 0}$ such that $F_1|_D, \ldots, F_m|_D$ are globally subanalytically prepared in x in a simultaneous way for every $D \in \mathcal{D}$. There is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is $D_C \in \mathcal{D}$ such that $(g(t), x) \in D_C$ for every $(t,x) \in C$. Fix a $C \in \mathcal{C}$, the globally subanalytic center ϑ, and for $j \in \{1, \ldots, m\}$ a preparing tuple $(0, y_0, a_j, q_j, s, v_j, b, P)$ for $F_j|_{D_C}$ where $y_0 := y - \vartheta(z)$ on D_C, $b := (b_1, \ldots, b_s)$, $P := (p_1, \ldots, p_s)^t$, and $a_1, \ldots, a_m, b_1, \ldots, b_s$ are globally subanalytic on $\pi^*(D_C)$. We have, for every $j \in \{1, \ldots, m\}$ and every $(z,x) \in D_C$,

$$F_j(z,x) = a_j(z)|x - \vartheta(z)|^{q_j} v_j(b_1(z)|x - \vartheta(z)|^{p_1}, \ldots, b_s(z)|x - \vartheta(z)|^{p_s}).$$

Let $h : C \to \mathbb{R}, (t,x) \mapsto x - \vartheta(g(t))$. Then it is immediately seen with Definition 2.14 that h is a 0–logarithmic scale. We obtain

$$f_j(t,x) = a_j(g(t))|h(t,x)|^{q_j} v_j(b_1(g(t))|h(t,x)|^{p_1}, \ldots, b_s(g(t))|h(t,x)|^{p_s})$$

for every $(t,x) \in C$ and are done, because

$$a_1(g), \ldots, a_m(g), \vartheta(g), b_1(g), \ldots, b_s(g) \colon \pi(C) \to \mathbb{R}$$

are C–nice by Remark 2.37. $\square$

Remark 2.56 *One may replace "nicely log-analytic" by "purely log-analytic" and "nicely 0–log-analytically prepared" by "purely 0–log-analytically prepared" in Proposition 2.55.*

Proposition 2.57 *Let $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f_1, \ldots, f_m : X \to \mathbb{R}$ be nicely log-analytic functions in x of order at most r. Then there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that $f_1|_C, \ldots, f_m|_C$ are nicely r–log-analytically prepared in a simultaneous way for every $C \in \mathcal{C}$.*

Proof We do an induction on r.

$r = 0$: Then $f_1, \ldots, f_m$ are nicely log-analytic in x of order 0 and we are done with Proposition 2.55.

$< r \to r$: It is enough to consider the following situation: Assume that there are $l, l' \in \mathbb{N}_0$, nicely log-analytic functions $g_1, \ldots, g_l : X \to \mathbb{R}$ and $h_1, \ldots, h_{l'} : X \to \mathbb{R}_{>0}$ in x of order at most $r-1$ and for every $j \in \{1, \ldots, m\}$ there is a globally subanalytic function $F_j : \mathbb{R}^{l+l'} \to \mathbb{R}$ such that

$$f_j = F_j(g_1, \ldots, g_l, \log(h_1), .., \log(h_{l'})).$$

Applying the inductive hypothesis to $g_1, \ldots, g_l, h_1, \ldots, h_{l'}$ and Corollary 2.47 we find a definable cell decomposition $\mathcal{U}$ of $X_{\neq 0}$ such that for every $U \in \mathcal{U}$ there is a nice $(r-1)$–logarithmic scale $\mathcal{Y}_{r-1} := (y_0, \ldots, y_{r-1})$ on U, a $k \in \mathbb{N}$, a U–nice function $\eta : \pi(U) \to \mathbb{R}^k$, and for every $j \in \{1, \ldots, m\}$ there is a globally subanalytic function $H_j : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}$ such that

$$f_j(t,x) = H_j(\eta(t), \mathcal{Y}_{r-1}(t,x), \log(|y_{r-1}(t,x)|))$$

for all $(t,x) \in U$. Fix $U \in \mathcal{U}$ and for this U a corresponding $\mathcal{Y}_{r-1}$, η, and H_j for $j \in \{1, \ldots, m\}$. By further decomposing U if necessary we may assume that either $|y_{r-1}| = 1$ or $|y_{r-1}| > 1$ or $|y_{r-1}| < 1$ on U. Assume the former. Then by (1) in Proposition 2.52 $f|_C$ is nicely log-analytic in x of order at most $(r-1)$ and we are done with the inductive hypothesis. Assume $|y_{r-1}| > 1$ or $|y_{r-1}| < 1$ on U. Then $\mathcal{Y} := (y_0, \ldots, y_r)$ is an r–logarithmic scale on U where $y_r := \log(|y_{r-1}|)$.

Let $(z,w) := (z_1, \ldots, z_k, w_0, \ldots, w_r)$ range over $\mathbb{R}^k \times \mathbb{R}^{r+1}$. Set $w' := (w_1, \ldots, w_r)$. Let $\pi^* : \mathbb{R}^k \times \mathbb{R}^{r+1} \to \mathbb{R}^{r+k}$ be the projection on $(z_1, \ldots, z_k, w_1, \ldots, w_r)$. With Fact 2.15 we find a globally subanalytic cell decomposition $\mathcal{D}$ of $\mathbb{R}^k \times \mathbb{R}^r \times \mathbb{R}_{\neq 0}$ such that $H_1|_D, \ldots, H_m|_D$ are globally subanalytically prepared in w_0 in a simultaneous way. There is a definable cell decomposition $\mathcal{A}$ of U such that for every $A \in \mathcal{A}$ there is $D_A \in \mathcal{D}$ such that $(\eta(t), \mathcal{Y}_r(t,x)) \in D_A$ for every $(t,x) \in A$. Fix $A \in \mathcal{A}$, the globally subanalytic center ϑ, and for $j \in \{1, \ldots, m\}$ an LA-preparing tuple $(0, y, a_j, q_j, s, v_j, b, P)$ for $F_j|_{D_A}$ where $y := w_0 - \vartheta(w', z)$ on D_A, $b := (b_1, \ldots, b_s)$, $P := (p_1, \ldots, p_s)^t$, and $a_1, \ldots, a_m, b_1, \ldots, b_s$ are globally subanalytic on $\pi^*(D_A)$. We have, for every $j \in \{1, \ldots, m\}$,

$$H_j|_{D_A} = a_j(z,w')|y(z,w)|^{q_j} v_j(b_1(z,w')|y(z,w)|^{p_1}, \ldots, b_s(z,w')|y(z,w)|^{p_s}).$$

From the inductive hypothesis we will derive the following two claims.

Claim 2.58 *Let $d \in \mathbb{N}$ and $p \in \{1, \ldots, r\}$. Let $\alpha_1, \ldots, \alpha_d : A^{\log(|y_0|)} \to \mathbb{R}$ be nicely log-analytic in x of order at most $p-1$. For $j \in \{1, \ldots, d\}$ consider*

$$\beta_j : A \to \mathbb{R}, (t,x) \mapsto \alpha_j(t, \log(|y_0(t,x)|)).$$

Then there is a definable cell decomposition $\mathcal{Q}$ of A such that for every $Q \in \mathcal{Q}$ the following hold.

(1) There are $\hat{\Theta}_1, \ldots, \hat{\Theta}_p : \pi(Q) \to \mathbb{R}$ such that $\beta_1|_Q, \ldots, \beta_d|_Q$ are nicely p–log-analytically prepared with center $\hat{\Theta} := (\Theta_0|_{\pi(Q)}, \hat{\Theta}_1, \ldots, \hat{\Theta}_p)$.

(2) Additionally, for every $j \in \{1, \ldots, d\}$ there is for $\beta_j|_Q$ a nice LA-preparing tuple $(p, \hat{\mathcal{Y}}_p, \hat{a}_j, \hat{q}_j, \hat{s}, \hat{v}_j, \hat{b}, \hat{P})$ such that $\hat{q}_j \in \{0\} \times \mathbb{Q}^p$ and $\hat{P} \in M_1(\hat{s} \times (p+1), \mathbb{Q})$ where $\hat{\mathcal{Y}}_p$ denotes the p–logarithmic scale with center $\hat{\Theta}$ on Q.

Proof Set $B := A^{\log(|y_0|)}$. Applying the inductive hypothesis to $\alpha_1, \ldots, \alpha_d$ we obtain a definable cell decomposition $\mathcal{S}$ of B such that $\alpha_1|_S, \ldots, \alpha_d|_S$ are nicely $(p-1)$–log-analytically prepared in a simultaneous way for every $S \in \mathcal{S}$. Consider the definable set

$$S^{\log(|y_0|)^*} := \{(t,x) \in A \mid (t, \log(|y_0(t,x)|)) \in S\}$$

for $S \in \mathcal{S}$. Then $\bigcup_{S \in \mathcal{S}} S^{\log(|y_0|)^*} = A$. Fix $S \in \mathcal{S}$ and the center $(\hat{\Theta}_1, \ldots, \hat{\Theta}_p)$ of $\alpha_1|_S, \ldots, \alpha_d|_S$. Let $T := S^{\log(|y_0|)^*}$. Note that $\beta_1|_T, \ldots, \beta_d|_T$ are nicely p–log-analytically prepared with center $(\Theta_0|_{\pi(T)}, \hat{\Theta}_1, \ldots, \hat{\Theta}_p)$ and for $j \in \{1, \ldots, d\}$ there is a nice LA-preparing tuple $(p, \hat{\mathcal{Y}}_{p,T}, \hat{a}_j, \hat{q}_j, \hat{s}, \hat{v}_j, \hat{b}, \hat{P})$ for $\beta_j|_T$ such that $\hat{q}_j \in \{0\} \times \mathbb{Q}^p$ and $\hat{P} \in M_1(\hat{s} \times (p+1), \mathbb{Q})$ by Remark 2.43 where $\hat{\mathcal{Y}}_{p,T}$ denotes the nice p–logarithmic scale with center $(\Theta_0|_{\pi(T)}, \hat{\Theta}_1, \ldots, \hat{\Theta}_p)$ on T. With the cell decomposition theorem applied to T we are done. □

Claim 2.59 *Let $\gamma : \mathbb{R}^k \times \mathbb{R}^r \to \mathbb{R}$ be globally subanalytic such that*

$$y_0 \sim_A \gamma(\eta, y_1, \ldots, y_r).$$

Then $f_j|_A$ is nicely log-analytic in x of order at most $r-1$ for every $j \in \{1, \ldots, m\}$.

Proof By (1) in Proposition 2.54 there is a nicely log-analytic function $\kappa\colon A^{\log(|y_0|)} \to \mathbb{R}$ in x of order at most $r-1$ such that $\gamma(\eta, y_1, \ldots, y_r) = \kappa(t, \log(|y_0|))$ on A. So by Claim 2.58 there is a definable cell decomposition $\mathcal{N}$ of A such that for every $N \in \mathcal{N}$ the function $\gamma(\eta, y_1, \ldots, y_r)|_N$ is nicely r–log-analytically prepared and there is a nice LA-preparing tuple $(r, \tilde{\mathcal{Y}}_r, \Psi, \tilde{q}, \tilde{s}, \tilde{v}, \tilde{b}, \tilde{P})$ for $\gamma(\eta, y_1, \ldots, y_r)|_N$ such that $\tilde{q} \in \{0\} \times \mathbb{Q}^r$ and $\tilde{P} \in M_1(\tilde{s} \times (r+1), \mathbb{Q})$ where $\tilde{\mathcal{Y}}_r$ has a center whose first component is $\Theta_0|_{\pi(N)}$. Fix $N \in \mathcal{N}$.

By Remark 2.18 it is enough to consider the following property $(*)_p$ for $p \in \{0, \ldots, r\}$ on N: There is a nice p–logarithmic scale $\tilde{\mathcal{Y}}_p := (y_0, \tilde{y}_1, \ldots, \tilde{y}_p)$ with center $(\Theta_0|_{\pi(N)}, \tilde{\Theta}_1, \ldots, \tilde{\Theta}_p)$, and an N–nice $\Psi\colon\ \pi(N) \to \mathbb{R}$ such that

$$y_0 \sim_N \Psi \prod_{l=1}^{p} |\tilde{y}_l|^{\tilde{q}_l}$$

where $\tilde{q}_l \in \mathbb{Q}$ for every $l \in \{1, \ldots, p\}$.

If $p = 0$ then $y_0 \sim_N \Psi$. We are done with (2) in Proposition 2.52 applied to $f_j|_N = H_j(\eta|_N, \mathcal{Y}|_N)$ for every $j \in \{1, \ldots, m\}$. Assume $p > 0$. By a suitable induction on p it is enough to establish the following statement: There is a decomposition $\mathcal{K}$ of N into finitely many definable sets such that for every $K \in \mathcal{K}$, either the function $f_j|_K$ is nicely log-analytic in x of order at most $r-1$ for every $j \in \{1, \ldots, m\}$, or $(*)_{p-1}$ holds on K.

For $M > 1$ let $N_{>M} := N_{>M}(\tilde{\mathcal{Y}}_p)$ and for $i \in \{1, \ldots, p\}$ let $N_{i,M} := N_{i,M}(\tilde{\mathcal{Y}}_p)$. By Remark 2.41 there is $M > 1$ and an N–nice $\xi : \pi(N) \to \mathbb{R}$ such that $y_0 \sim_{N_{>M}} \xi$. Fix such an M. Since $N = N_{>M} \cup N_{1,M} \cup \ldots \cup N_{p,M}$, it suffices to establish the statement for $N_{>M}$ and $N_{i,M}$ instead of for N, where $i \in \{1, \ldots, p\}$.

$B := N_{>M}$: By (2) in Remark 2.36 $\xi|_B$ is B–nice. Therefore with (2) in Proposition 2.52 applied to $f_j|_B = H_j(\eta|_B, \mathcal{Y}_r|_B)$ we obtain that $f_j|_B$ is a nicely log-analytic function in x of order at most $r-1$.

$B_i := N_{i,M}$ for $i \in \{1, \ldots, p\}$: Set $\tilde{y}_0 := y_0$. It holds $|\tilde{y}_i| < M$ on B_i. So we obtain

$$\frac{1}{\delta} < \frac{|\tilde{y}_{i-1}|}{\exp(\tilde{\Theta}_i)} < \delta$$

for $\delta := e^M$ on B_i which gives $\tilde{y}_{i-1} \sim_{B_i} \exp(\tilde{\Theta}_i)|_{\pi(B_i)}$.

Assume $i = 1$. With Remark 2.35 we see that $\exp(\tilde{\Theta}_1)|_{\pi(B_1)}$ is a B_1–nice B_1–heir. Again with (2) in Proposition 2.52 applied to $f_j|_{B_1} = H_j(\eta|_{B_1}, \mathcal{Y}|_{B_1})$ (we have $y_0 \sim_{B_1} \xi$ with $\xi = \exp(\tilde{\Theta}_1)|_{\pi(B_1)}$) we see that $f_j|_{B_1}$ is nicely log-analytic in x of order at most $r-1$ for every $j \in \{1, \ldots, m\}$.

Assume $i > 1$. Then $p > 1$. Let

$$\Xi : N \to \mathbb{R}, \quad (t,x) \mapsto \Psi(t) \prod_{l=1}^{p} |\tilde{y}_l(t,x)|^{\tilde{q}_l}.$$

By (the proof of) (1) in Proposition 2.46 there is a globally subanalytic function $G : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ such that $\Xi = G(\Psi, \tilde{y}_1, \ldots, \tilde{y}_p)$ on B_i. By (2) in Proposition 2.54

there is a nicely log-analytic function $\lambda : B_i^{\log(|y_0|)} \to \mathbb{R}$ in x of order at most $p-2$ such that $\Xi(t,x) = \lambda(t, \log(|y_0(t,x)|))$ for every $(t,x) \in B_i$. Note that $0 \leq p-2 < r$. With Claim 2.58 applied to Ξ we find a definable cell decomposition $\mathcal{K}$ of B_i such that $\Xi|_K$ is nicely $(p-1)$–log-analytically prepared with nice LA-preparing tuple $(p-1, \mathcal{Y}_{p-1}, \bar{a}_j, \bar{q}_j, \bar{s}, \bar{v}_j, \bar{b}, \bar{P})$ where $\bar{q}_j \in \{0\} \times \mathbb{Q}^{p-1}$, $\bar{P} \in M_1(\bar{s} \times p, \mathbb{Q})$, and $\mathcal{Y}_{p-1}$ is a nice $(p-1)$–logarithmic scale with center $\bar{\Theta} := (\Theta_0|_{\pi(K)}, \bar{\Theta}_1, \ldots, \bar{\Theta}_{p-1})$. With Remark 2.18 we obtain property $(*)_{p-1}$ applied to every $K \in \mathcal{K}$. $\square$

Case 1: $\vartheta = 0$. Let $a_{m+j} := b_j$ for $j \in \{1, \ldots, s\}$. By (1) in Proposition 2.54 there are nicely log-analytic functions $\alpha_1, \ldots, \alpha_{m+s} : A^{\log(|y_0|)} \to \mathbb{R}$ in x of order at most $r-1$ such that for every $(t,x) \in A$ we have

$$a_j(\eta(t), y_1(t,x), \ldots, y_r(t,x)) = \alpha_j(t, \log(|y_0(t,x)|))$$

for every $j \in \{1, \ldots, m+s\}$. With Claim 2.58 applied to

$$\beta_j : A \to \mathbb{R}, \quad (t,x) \mapsto \alpha_j(t, \log(|y_0(t,x)|)),$$

for $j \in \{1, \ldots, m+s\}$ we are done by using composition of power series.

Case 2: $\vartheta \neq 0$. There is $\epsilon \in \,]0,1[$ such that $0 < |w_0 - \vartheta(z,w')| < \epsilon|w_0|$ for $(z,w) \in D_A$. This gives with Remark 2.12 $w_0 \sim_{D_A} \vartheta(z,w')$ and therefore

$$y_0 \sim_A \vartheta(\eta, y_1, \ldots, y_r).$$

By Claim 2.59 $f_j|_A$ is a nicely log-analytic function in x of order at most $r-1$ for every $j \in \{1, \ldots, m\}$. With the inductive hypothesis applied to $f_j|_A$ for $j \in \{1, \ldots, m\}$ we are done. $\square$

Since every log-analytic function is nicely log-analytic in x, Theorem 0.2 is established.

Theorem 2.60 *Let $m \in \mathbb{N}$, $r \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f_1, \ldots, f_m : X \to \mathbb{R}$ be log-analytic functions of order at most r. Then there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that $f_1|_C, \ldots, f_m|_C$ are nicely r–log-analytically prepared in x in a simultaneous way for every $C \in \mathcal{C}$.*

In light of Theorem 2.60 one may ask if there is an another kind of preparation of log-analytic functions of order > 0 with log-analytic data only. To investigate this question one notes that an interesting example is the class of constructible functions introduced by Cluckers and Miller in [1]. A function $f : \mathbb{R}^m \to \mathbb{R}$ is called constructible if it is a finite sum of finite products of functions on $\mathbb{R}^m$ which are definable in $\mathbb{R}_{\text{an}}$ and of functions which are the logarithm of a positive function on $\mathbb{R}^m$ which is definable in $\mathbb{R}_{\mathrm{an}}$. This is a proper larger class than the globally subanalytic functions, but a proper subclass of the class of log-analytic ones of order at most 1 (since the function $\mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{\log(|x|+1)}$, is log-analytic of order 1 but not constructible). For constructible functions there is a pure preparation not in terms of units but suitable for questions on integration (compare with [1]).

A important consequence of Theorem 2.60 is the following. Call a definable cell $C \subset \mathbb{R}^n \times \mathbb{R}$ **simple** if C_t is of the form $]0, d_t[$ for every $t \in \pi(C)$ where $d_t \in \mathbb{R}_{>0} \cup \{\infty\}$ (see for example Definition 2.15 in Kaiser–Opris [8]). Proposition 2.19 in [8] states that the center of every logarithmic scale on such a simple cell C vanishes. This gives the following (compare with Theorem 2.30 in [8]).

Corollary 2.61 *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f : X \to \mathbb{R}$ be log-analytic of order at most r. Then there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ we have that $f|_C$ is purely r–log-analytically prepared in x with center 0.*

Proof By Theorem 2.60 there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ we have that $f|_C$ is nicely r–log-analytically prepared in x. Fix a simple $C \in \mathcal{C}$ and let $(r, \mathcal{Y}, a, q, s, v, b, P)$ be a nice LA-preparing tuple for f. Let Θ be the center of $\mathcal{Y}$. Note that $\Theta = 0$. Let E be a set of C–heirs such that a and b can be constructed from E. Let $h \in E$. There is $\hat{r} \in \mathbb{N}_0$, an $\hat{r}$–logarithmic scale $\hat{\mathcal{Y}}$ with center $(\hat{\Theta}_0, \ldots, \hat{\Theta}_{\hat{r}})$ on C and $l \in \{1, \ldots, \hat{r}\}$ such that $h = \exp(\hat{\Theta}_l)$. Note that $\hat{\Theta}_l = 0$. So we have $h = 1$. So we obtain $E = \emptyset$ or $E = \{1\}$. With the proof of Claim 2.34 (see (2) in Example 2.33) one sees that a and b are log-analytic. $\square$

Important consequences of Corollary 2.61 are differentiability results for the class of log-analytic functions like strong quasianalyticity and a parametric version of Tamm's Theorem (see [8]).

3 Preparation theorems for definable functions in $\mathbb{R}_{\mathrm{an,exp}}$

This section is devoted to the proofs of Theorem 0.4 and C, the latter being the main result of the paper. We start with some preparations which we need for the rigorous proof of this both theorems.

3.1 Preparations

For Section 3.1 we fix $m \in \mathbb{N}$, a tuple of variables $v := (v_1, \ldots, v_m)$, and a definable set $X \subset \mathbb{R}^m$. Fix $k, l \in \mathbb{N}_0$ and definable functions $g_1, \ldots, g_k : X \to \mathbb{R}$ and $h_1, \ldots, h_l : X \to \mathbb{R}$. Set $\beta := (g, h_l, \exp(h_1), \ldots, \exp(h_l))$. (We could also write $\beta = (g, \exp(h_1), \ldots, \exp(h_l))$, but this simplifies notation below.) Note that $\beta(X) \subset \mathbb{R}^{k+l} \times \mathbb{R}_{>0}$. So there exists a 0–logarithmic scale on $\beta(X)$, ie $\beta(X)$ is 0-admissible.

Fix a log-analytic function $F : \mathbb{R}^{k+l+1} \to \mathbb{R}$. Let $\alpha : X \to \mathbb{R}, \quad v \mapsto F(\beta(v))$. Let $y := (y_1, \ldots, y_{k+l})$ range over $\mathbb{R}^{k+l}$. Let z be another single variable such that (y, z) ranges over $\mathbb{R}^{k+l} \times \mathbb{R}$. Let $\pi^* : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}^{k+l}, \quad (y, z) \mapsto y$.

Proposition 3.1 *Let $\Theta : \mathbb{R}^{k+l} \to \mathbb{R}$ be log-analytic such that*

$$\exp(h_l) \sim_X \Theta(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})).$$

There is a log-analytic function $G : \mathbb{R}^{k+l} \to \mathbb{R}$ such that

$$\alpha = G(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1}))$$

on X.

Proof Let $\kappa := \Theta(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1}))$. Note that $\kappa > 0$. There is $\delta > 1$ such that

$$\frac{1}{\delta} < \frac{\exp(h_l)}{\kappa} < \delta$$

on X. By taking logarithm we get, with $\lambda := \log(\delta)$,

$$-\lambda < h_l - \log(\kappa) < \lambda$$

on X. Set

$$\exp^* : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} \exp(x), & x \in [-\lambda, \lambda], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\exp^*$ is globally subanalytic and we have

$$\alpha = F(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1}), \kappa \cdot \exp^*(h_l - \log(\kappa)))$$

on X. Consider

$$G : \mathbb{R}^{k+l} \to \mathbb{R}, \qquad y \mapsto \begin{cases} F(y, \Theta(y) \cdot \exp^*(y_{k+1} - \log(\Theta(y)))), & y \in \pi^*(\beta(X)), \\ 0, & \text{otherwise.} \end{cases}$$

Note that G is well-defined, log-analytic and we obtain

$$\alpha = G(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1}))$$

on X since for $x \in X$ we have

$$\big(g(x), h_l(x), \exp(h_1(x)), \ldots, \exp(h_{l-1}(x))\big) \in \pi^*(\beta(X)). \qquad \square$$

Proposition 3.2 *Assume that F is positive and purely 0–log-analytically prepared in z with center 0 on $\beta(X)$. Then there is a purely log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ in z of order 0 such that*

$$\log(F(\beta)) = H(\beta)$$

on X.

Proof Let $(0, \mathcal{Y}, a, q, s, v, b, P)$ be a pure LA-preparing tuple for f where $b := (b_1, \ldots, b_s)$ and $\mathcal{Y} := y$. Note that $a > 0$. Consider

$$\eta := (\eta_0, \ldots, \eta_s) : \pi^*(\beta(X)) \to \mathbb{R}_{>0} \times \mathbb{R}^s, \quad y \mapsto (a(y), b_1(y), \ldots, b_s(y)).$$

Note that η is log-analytic. Let $w := (w_0, \ldots, w_{s+2})$ range over $\mathbb{R}^{s+3}$. For $w \in \mathbb{R}_{>0} \times \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}_{\neq 0}$ with $-1 \leq w_i |w_{s+2}|^{p_{0i}} \leq 1$ for every $i \in \{1, \ldots, s\}$ let

$$\phi(w) := \log(w_0) + q w_{s+1} + \log\left(v(w_1 |w_{s+2}|^{p_{01}}, \ldots, w_s |w_{s+2}|^{p_{0s}})\right).$$

Consider

$$G : \mathbb{R}_{>0} \times \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}_{\neq 0} \to \mathbb{R},$$
$$w \mapsto \begin{cases} \phi(w), & -1 \leq w_i |w_{s+2}|^{p_{0i}} \leq 1 \text{ for every } i \in \{1, \ldots, s\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then G is purely log-analytic in w_{s+2} of order 0 since $\log(v)$ is globally subanalytic. Note that

$$\log(F(\beta)) = G(\eta(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})), h_l, \exp(h_l))$$

on X. Then

$$H : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}, \quad (y, z) \mapsto \begin{cases} G(\eta(y_1, \ldots, y_{k+l}), y_{k+1}, z), & (y, z) \in \beta(X), \\ 0, & \text{otherwise,} \end{cases}$$

does the job, because H is purely log-analytic in z of order 0. □

Corollary 3.3 *Let $c, d \in \mathbb{N}$. Suppose there are functions $\mu_1, \ldots, \mu_c : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ and $\nu_1, \ldots, \nu_d : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}_{>0}$ which are purely 0–log-analytically prepared in z with center Θ on $\beta(X)$, and a globally subanalytic function $G : \mathbb{R}^{c+d} \to \mathbb{R}$ such that*

$$\alpha = G\left(\mu_1(\beta), \ldots, \mu_c(\beta), \log(\nu_1(\beta)), \ldots, \log(\nu_d(\beta))\right).$$

If $\Theta = 0$ there is a purely log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ in z of order 0 such that $\alpha = H(\beta)$ on X. If $\Theta \neq 0$ then we have

$$\exp(h_l) \sim_X \Theta(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})).$$

Proof Note that $y_0 : \beta(X) \to \mathbb{R}, (y,z) \mapsto z - \Theta(y)$, is a 0–logarithmic scale with log-analytic center Θ.

Case 1: $\Theta = 0$. Then by Proposition 3.2 there are purely log-analytic functions $H_1, \ldots, H_m : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ in z of order 0 such that $\log(\nu_j(\beta)) = H_j(\beta)$ on X for every $j \in \{1, \ldots, m\}$. By (2) in Remark 2.27 we see that $\mu_1, \ldots, \mu_c$ are purely log-analytic in z of order 0. Because G is globally subanalytic we are done.

Case 2: $\Theta \neq 0$. By Remark 2.12 it is $z \sim_{\beta(X)} \Theta$. We obtain the result. □

Corollary 3.4 *Let $r \in \mathbb{N}_0$. Suppose F is purely log-analytic in z of order r. Then there is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for $C \in \mathcal{C}$ the following holds. There is a purely log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ in z of order 0 such that $\alpha|_C = H(\beta|_C)$ or there is a log-analytic function $\Theta : \mathbb{R}^{k+l} \to \mathbb{R}$ such that*

$$\exp(h_l) \sim_C \Theta(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})).$$

Proof We do an induction on r.
$r = 0$: Then F is purely log-analytic in z of order 0. The assertion follows.
$r-1 \to r$: With Definition 2.25 it is enough to consider the following situation. Let $c, d \in \mathbb{N}_0$, a globally subanalytic function $G : \mathbb{R}^{c+d} \to \mathbb{R}$, purely log-analytic functions $\rho_1, \ldots, \rho_c : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ and $\sigma_1, \ldots, \sigma_d : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}_{>0}$ in z of order at most $r-1$ be such that $\alpha = G(\rho_1(\beta), \ldots, \rho_c(\beta), \log(\sigma_1(\beta)), \ldots, \log(\sigma_d(\beta)))$. Applying the inductive hypothesis to $\rho_1(\beta), \ldots, \rho_c(\beta), \sigma_1(\beta), \ldots, \sigma_d(\beta)$ there is a decomposition $\mathcal{A}$ of X into finitely many definable cells such that the following holds for $A \in \mathcal{A}$. There are functions $\mu_1, \ldots, \mu_c : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ and $\nu_1, \ldots, \nu_d : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}_{>0}$ which are purely log-analytic in z of order 0 such that

$$\alpha = G(\mu_1(\beta), \ldots, \mu_c(\beta), \log(\nu_1(\beta)), \ldots, \log(\nu_d(\beta)))$$

on A (G does not change) or there is a log-analytic function $\tilde{\Theta} : \mathbb{R}^{k+l} \to \mathbb{R}$ such that

$$\exp(h_l) \sim_A \tilde{\Theta}(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})).$$

Fix such an A. If the latter holds we are done. So assume the former. Fix the corresponding $\mu_1, \ldots, \mu_c, \nu_1, \ldots, \nu_d$. By Remark 2.56 there is a definable cell decomposition $\mathcal{D}$ of $\mathbb{R}^{k+l} \times \mathbb{R}_{\neq 0}$ such that $\mu_1, \ldots, \mu_c, \nu_1, \ldots, \nu_d$ are purely 0–log-analytically prepared in z in a simultaneous way on every $D \in \mathcal{D}$. There is a definable cell decomposition $\mathcal{C}$ of A such that for every $C \in \mathcal{C}$ there is $D_C \in \mathcal{D}$ with $\beta(C) \subset D_C$. Fix $C \in \mathcal{C}$ and the center Θ of the pure 0–preparation of $\mu_1, \ldots, \mu_c, \nu_1, \ldots, \nu_d$ on D_C. If $\Theta = 0$ we find with Corollary 3.3 a purely log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ in z of order 0 such that $\alpha|_C = H(\beta|_C)$. If $\Theta \neq 0$ then with Remark 2.12 $z \sim_{D_C} \Theta(y)$ and therefore

$$\exp(h_l) \sim_C \Theta(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})).$$ □

3.2 Proof of Theorem 0.4 and Theorem 0.5

For a definable function $f : X \to \mathbb{R}$ (where $X \subset \mathbb{R}^m$ is definable) there exist $e \in \mathbb{N}_0$ and a set E of positive definable functions on X such that f has exponential number at most e with respect to E (ie is a composition of log-analytic functions and exponentials from E, see Definition 1.5 and Remark 1.7).

The first goal for this section is to prepare f cellwise as a product of a log-analytic function, an exponential of a function h which is the restriction of a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$ with exponential number at most $e-1$ with respect to E, and a unit of a special form (see Theorem 0.4). Additionally h is itself prepared. To be more precise we introduce cellwise a set P of positive definable functions (or exponentials) which occur in the preparation of f and show that every function from $\log(P)$ is also prepared (ie a product of a log-analytic function, an exponential of a function g with $\exp(g) \in P$ and a unit of a special form). So analytical properties of functions from $\log(E)$ closed under taking finite $\mathbb{Q}$–linear combinations (like the property of being locally bounded) can be cellwise transferred to every function in $\log(P)$ (see Opris [12]).

The second goal for this section is to prove Theorem 0.5 by combining Theorem 2.60 and Theorem 0.4: From Theorem 0.4 we also get cellwise a finite set L of log-analytic functions which occur in the preparation of f and with Theorem 2.60 we prepare all functions from L simultaneously.

Definition 3.5 *Let $X \subset \mathbb{R}^m$ be definable and $f : X \to \mathbb{R}$ be a function. Let E be a set of positive definable functions on X.*

(1) *Let L be a set of log-analytic functions on X. By induction on $e \in \mathbb{N}_0 \cup \{-1\}$ we define that $f : X \to \mathbb{R}$ is **e–prepared with respect to L and E**.*
Base Case. *The function f is (-1)–prepared with respect to L and E if f is the zero function.*
Inductive step. *The function f is e–prepared with respect to L and E if*

$$f = a \cdot \exp(c) \cdot u$$

where $a \in L$ vanishes or does not have a zero, $c : X \to \mathbb{R}$ is $(e-1)$–prepared with respect to L and E, $\exp(c) \in E$ and $u : X \to \mathbb{R}$ is a function of the form

$$u = v(b_1 \cdot \exp(d_1), \ldots, b_s \cdot \exp(d_s))$$

where $s \in \mathbb{N}_0$, $b_j \in L$ does not have any zero, $d_j : X \to \mathbb{R}$ is $(e-1)$–prepared with respect to L and E and $\exp(d_j) \in E$ for every $j \in \{1, \ldots, s\}$. Additionally,

v is a power series which converges absolutely on an open neighborhood of $[-1,1]^s$, it is $b_j(x)\exp(d_j(x)) \in [-1,1]$ for every $x \in X$ and every $j \in \{1,\ldots,s\}$, and $v([-1,1]^s) \subset \mathbb{R}_{>0}$.

(2) We say that f is e**–prepared with respect to** E if there is a set L of log-analytic functions on X such that f is e–prepared with respect to L and E.

Remark 3.6 *Let $X \subset \mathbb{R}^m$ be definable. The following hold.*

(1) *Let $e \geq 0$. If $f : X \to \mathbb{R}$ is e–prepared with respect to a set E of positive definable functions then $1 \in E$.*
(2) *Let $f : X \to \mathbb{R}$ be log-analytic. Then f is 0–prepared with respect to $L := \{f\}$ and $E := \{1\}$.*
(3) *If f is 0–prepared with respect to a set E of positive definable functions then f is log-analytic.*

Remark 3.7 *Let $X \subset \mathbb{R}^m$ be definable. Let E be a set of positive definable functions and L be a set of log-analytic functions on X. Let $e \in \mathbb{N}_0$. Let $f : X \to \mathbb{R}$ be e–prepared with respect to L and E. There are $k \in \mathbb{N}$, log-analytic functions $h_1,\ldots,h_k \in L$, $(e-1)$–prepared $g_1,\ldots,g_k$ with respect to L and E with $\exp(g_1),\ldots,\exp(g_k) \in E$ and a globally subanalytic function $G : \mathbb{R}^{2k} \to \mathbb{R}$ such that*

$$f = G(h_1,\ldots,h_k,\exp(g_1),\ldots,\exp(g_k)).$$

Proof The proof is similar as the proof of (1) in Proposition 2.46. □

Remark 3.8 *Let $X \subset \mathbb{R}^m$ be definable. Let E be a set of positive definable functions on X. Let $e \in \mathbb{N}_0$. Let $f : X \to \mathbb{R}$ be e–prepared with respect to E. Then f has exponential number at most e with respect to E and is therefore definable.*

Proof This is easily seen with Remark 3.7 and an easy induction on e. □

Now we are ready to formulate and prove Theorem 0.4.

Theorem 3.9 *Let $X \subset \mathbb{R}^m$ be definable and $e \in \mathbb{N}_0$. Let $f : X \to \mathbb{R}$ be a function. Let E be a set of positive definable functions on X such that f has exponential number at most e with respect to E. Then there is a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for every $C \in \mathcal{C}$ there is a finite set P of positive definable functions on C and a finite set L of log-analytic functions on C such that the following hold.*

(1) *$f|_C$ is e–prepared with respect to L and P and for every $g \in \log(P)$ there is $\alpha \in \{-1, \ldots, e-1\}$ such that g is α–prepared with respect to L and P.*
(2) *P satisfies the following condition $(*_e)$ with respect to $E|_C$: If $g \in \log(P)$ is l–prepared with respect to L and P for $l \in \{0, \ldots, e-1\}$ then g is a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$ restricted to C which have exponential number at most l with respect to E.*

Proof We do an induction on e.
$e = 0$: Then f is log-analytic and we are done by choosing $P = \{1\}$ and $L = \{f\}$.
$e-1 \to e$: There are $k, l \in \mathbb{N}$, functions $g_1, \ldots, g_k, h_1, \ldots, h_l : \mathbb{R}^m \to \mathbb{R}$ with exponential number at most $e-1$ with respect to E, and a log-analytic function $F : \mathbb{R}^{k+l} \to \mathbb{R}$ such that $f = F(g_1, \ldots, g_k, \exp(h_1), \ldots, \exp(h_l))$ and $\exp(h_1), \ldots, \exp(h_l) \in E$.

By an auxiliary induction on l and involving the inductive hypothesis we may assume that the theorem is proven for functions of the form

$$\kappa = H(g_1, \ldots, g_k, h_l, \exp(h_1), \ldots, \exp(h_{l-1}))$$

on X where $H : \mathbb{R}^{k+l} \to \mathbb{R}$ is log-analytic. $(**)$

(If $l = 1$ then $(**)$ holds by the inductive hypothesis: $g = H(g_1, \ldots, g_k, h_l)$ has exponential number at most $e-1$ with respect to E by Remark 1.8.)

(This includes also functions of the form $\kappa = \hat{H}(g_1, \ldots, g_k, \exp(h_1), \ldots, \exp(h_{l-1}))$ on X where $\hat{H} : \mathbb{R}^{k+l-1} \to \mathbb{R}$ is log-analytic.)

Let $g := (g_1, \ldots, g_k)$. Let $y := (y_1, \ldots, y_{k+l})$ range over $\mathbb{R}^{k+l}$. Let z be another single variable such that (y, z) ranges over $\mathbb{R}^{k+l} \times \mathbb{R}$. Let $y' := (y_1, \ldots, y_{k+l-1})$. Let $\pi^+ : \mathbb{R}^{k+l} \to \mathbb{R}^{k+l-1}, \quad y \mapsto y'$. Let $r \in \mathbb{N}_0$ be such that F is purely log-analytic in y_{k+l} of order at most r.

Case 1: $r = 0$. By Remark 2.56 we find a definable cell decomposition $\mathcal{D}$ of $\mathbb{R}^{k+l-1} \times \mathbb{R}_{\neq 0}$ such that $F|_D$ is purely 0–log-analytically prepared in y_{k+l} for every $D \in \mathcal{D}$. There is a decomposition $\mathcal{A}$ of X into finitely many definable cells such that for every $A \in \mathcal{A}$ there is $D_A \in \mathcal{D}$ such that for every $x \in A$ we have $(g(x), \exp(h_1(x)), \ldots, \exp(h_l(x))) \in D_A$. Fix $A \in \mathcal{A}$ and a purely preparing tuple $(0, \mathcal{Y}, a, q, s, v, b, P)$ for $F|_{D_A}$ where $b := (b_1, \ldots, b_s)$ and $P := (p_1, \ldots, p_s)^t$. Let Θ be the center of $\mathcal{Y}$. Then one of the following properties holds.

(1) There is $\epsilon \in]0, 1[$ such that $0 < |y_{k+l} - \Theta(y')| < \epsilon |y_{k+l}|$ for all $y \in D_A$.
(2) $\Theta = 0$ on $\pi^+(D_A)$.

Assume (1). Then by Remark 2.12 we have $y_{k+l} \sim_{D_A} \Theta$. This gives

$$\exp(h_l) \sim_A \Theta(g, \exp(h_1), \ldots, \exp(h_{l-1})).$$

By Proposition 3.1 there is a log-analytic function $G : \mathbb{R}^{k+l} \to \mathbb{R}$ such that

$$f = G(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1}))$$

on A. With $(**)$ applied to f we are done.

Assume (2). Let $\beta := (g, \exp(h_1), \ldots, \exp(h_{l-1}))$. It holds

$$f|_A = a(\beta)\exp(qh_l)v(b_1(\beta)\exp(p_1h_l), \ldots, b_s(\beta)\exp(p_sh_l)).$$

Let $b_0 := a$. Note that we can apply $(**)$ to $b_j(\beta)$ for $j \in \{0, \ldots, s\}$ and so there is a decomposition $\mathcal{B}$ of A into finitely many definable cells such that for every $B \in \mathcal{B}$ the following holds: There is a finite set P' of positive definable functions on B and a finite set L' of log-analytic functions on B such that $b_0(\beta)|_B, \ldots, b_s(\beta)|_B$ are e–prepared with respect to L' and P' and that P' satisfies property $(*_e)$ with respect to $E|_B$. Fix such a B. Then we have, for $j \in \{0, \ldots, s\}$,

$$b_j(\beta) = \hat{a}_j\exp(\hat{c}_j)\hat{v}_j(\hat{b}_{1j}\exp(\hat{d}_{1j}), \ldots, \hat{b}_{s_jj}\exp(\hat{d}_{s_jj}))$$

where $s_j \in \mathbb{N}$, $\hat{a}_j, \hat{b}_{1j}, \ldots, \hat{b}_{s_jj}$ are log-analytic and $\hat{c}_j, \hat{d}_{1j}, \ldots, \hat{d}_{s_jj}$ are finite $\mathbb{Q}$–linear combinations of functions from $\log(E)$ restricted to B which have exponential number at most $e-1$ with respect to E and $\hat{v}_j$ is a power series which converges absolutely on an open neighborhood of $[-1,1]^{s_j}$ and $\hat{v}_j([-1,1]^{s_j}) \subset \mathbb{R}_{>0}$. Since $h_l \in \log(E)$ has exponential number at most $e-1$ with respect to E we obtain that $\hat{c}_0 + qh_l$ and $\hat{c}_j + p_jh_l$ for $j \in \{1, \ldots, s\}$ are finite $\mathbb{Q}$–linear combinations of functions from $\log(E)$ restricted to B which have exponential number at most $e-1$ with respect to E. (Note that h_l is not necessarily $(e-1)$–prepared with respect to P' on B.)

Consequently we obtain by composition of power series that there are $\tilde{s} \in \mathbb{N}$, log-analytic functions $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_{\tilde{s}} : B \to \mathbb{R}$, and functions $c, d_1, \ldots, d_{\tilde{s}} : B \to \mathbb{R}$ which are finite $\mathbb{Q}$–linear combinations of elements from $\log(E)$ restricted to B which have exponential number at most $e-1$ with respect to $E|_B$ such that the following holds: we have

$$f|_B = \tilde{a}\exp(c)\tilde{v}(\tilde{b}_1\exp(d_1), \ldots, \tilde{b}_{\tilde{s}}\exp(d_{\tilde{s}}))$$

where $\tilde{b}_j(x)\exp(d_j(x)) \in [-1,1]$ for every $x \in B$ and $j \in \{1, \ldots, \tilde{s}\}$, and $\tilde{v}$ is a power series which converges absolutely on an open neighborhood of $[-1,1]^{\tilde{s}}$ and $\tilde{v}([-1,1]^{\tilde{s}}) \subset \mathbb{R}_{>0}$. Note that c and $d_1, \ldots, d_{\tilde{s}}$ have exponential number at most $e-1$ with respect to E. By the inductive hypothesis there is a further decomposition $\mathcal{C}_B$ of B into finitely many definable cells such that for every $C \in \mathcal{C}_B$ there is a finite set $\tilde{P}$

of positive definable functions on C which satisfies property $(*_{e-1})$ with respect to $E|_C$ and a finite set $\tilde{L}$ of log-analytic functions on C such that the functions $c|_C$ and $d_1|_C, \ldots, d_{\tilde{s}}|_C$ are $(e-1)$–prepared with respect to $\tilde{L}$ and $\tilde{P}$. So choose

$$L := \tilde{L} \cup \{\tilde{a}|_C, \tilde{b}_1|_C, \ldots, \tilde{b}_{\tilde{s}}|_C\}$$

and
$$P := \tilde{P} \cup \{\exp(c)|_C, \exp(d_1)|_C, \ldots, \exp(d_{\tilde{s}})|_C\}.$$

Then P satisfies property $(*_e)$ with respect to $E|_C$. Note that $f|_C$ is e–prepared with respect to L and P and we are done.

Case 2: $r > 0$. By Corollary 3.4 there is a decomposition $\mathcal{A}$ of X into finitely many definable cells such that for every $A \in \mathcal{A}$ one of the following properties holds.

(1) There is a purely log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \to \mathbb{R}$ in z of order 0 such that $f|_A = H(g, h_l, \exp(h_1), \ldots, \exp(h_l))$.

(2) There is a log-analytic function $\tilde{\Theta} : \mathbb{R}^{k+l} \to \mathbb{R}$ such that

$$\exp(h_l) \sim_A \tilde{\Theta}(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1})).$$

Let $A \in \mathcal{A}$. If (1) holds for A then we are done with Case 1. If (2) holds for A then by Proposition 3.1 there is a log-analytic function $H : \mathbb{R}^{k+l} \to \mathbb{R}$ such that

$$f = H(g, h_l, \exp(h_1), \ldots, \exp(h_{l-1}))$$

on A. We are done with $(**)$ applied to f. $\square$

In the following we aim for the second goal of Section 3.2, the proof of Theorem 0.5. In contrast to the treatment of Lion–Rolin in [9] we describe precisely how the unit in the preparation in Theorem 0.5 looks like, bound the number of iterations of the exponentials which occur in the preparation by the exponential number of f and determine which exponential functions occur: From Theorem 3.9 we cellwise take over the set P of exponentials (each function from $\log(P)$ is a restriction of a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$) and Theorem 2.60, the log-analytic preparation theorem, gives C–nice data (ie C–heirs).

Furthermore the preparation of every $g \in \log(P) \cup \{f\}$ is described by a pair $(l, r) \in (\mathbb{N}_0 \cup \{-1\}) \times \mathbb{N}_0$: From Theorem B we obtain finite sets L,P and an $l \in \mathbb{N}_0 \cup \{-1\}$ such that g is l–prepared with respect to L and P and from Theorem 2.60 an $r \in \mathbb{N}_0$ such that the functions from L are nicely r–log-analytically prepared in a simultaneous way (the number r does not depend on g and is chosen so that every function from L is log-analytic of order at most r).

After the complete proof of Theorem 0.5 we will give some important consequences.

For the rest of this section let $n \in \mathbb{N}_0$, $t := (t_1, \ldots, t_n)$ range over $\mathbb{R}^n$, x over $\mathbb{R}$ and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n, (t, x) \mapsto t$.

Definition 3.10 *Let $r \in \mathbb{N}_0$ and $e \in \mathbb{N}_0 \cup \{-1\}$. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable and $f : C \to \mathbb{R}$ be a function. Let E be a set of positive definable functions on C. By induction on e we define that f is (e, r)***–prepared** *in x with center Θ with respect to E. To this preparation we assign a* ***preparing tuple*** *for f.*

$e = -1$: We say that f is $(-1, r)$–prepared in x with center Θ with respect to E if f is the zero function. A preparing tuple for f is (0).

$e - 1 \to e$: We say that f is (e, r)–prepared in x with center Θ with respect to E if

$$f(t,x) = a(t)|\mathcal{Y}(t,x)|^{\otimes q} e^{c(t,x)} u(t,x)$$

for every $(t, x) \in C$ where $a : \pi(C) \to \mathbb{R}$ is C–nice which vanishes identically or has no zero, $\mathcal{Y} := (y_0, \ldots, y_r)$ is a nice r–logarithmic scale with center Θ, $q = (q_0, \ldots, q_r) \in \mathbb{Q}^{r+1}$, $\exp(c) \in E$ where c is $(e-1, r)$–prepared in x with center Θ with respect to E and $u = v \circ \phi$ where v is a power series which converges on an open neighborhood of $[-1, 1]^s$ with $v([-1, 1]^s) \subset \mathbb{R}_{>0}$ and $\phi := (\phi_1, \ldots, \phi_s) : C \to [-1, 1]^s$ is a function of the form

$$\phi_j := b_j(t)|y_0|^{p_{j0}} \cdot \ldots \cdot |y_r|^{p_{jr}} \exp(c_j(t,x))$$

for $j \in \{1, \ldots, s\}$ where $b_j : \pi(C) \to \mathbb{R}$ is C–nice, $p_{j0}, \ldots, p_{jr} \in \mathbb{Q}$, $\exp(c_j) \in E$ and c_j is $(e-1, r)$–prepared in x with center Θ with respect to E. A preparing tuple for f is then

$$\mathcal{J} := (r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

where $b := (b_1, \ldots, b_s)$, $\exp(d) := (\exp(d_1), \ldots, \exp(d_s))$ and

$$P := \begin{pmatrix} p_{10} & \cdot & \cdot & p_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{s0} & \cdot & \cdot & p_{sr} \end{pmatrix} \in M\big(s \times (r+1), \mathbb{Q}\big).$$

Remark 3.11 *Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let E be a set of positive definable functions on C. Let $f : C \to \mathbb{R}$ be (e, r)–prepared in x with respect to E. If $e = 0$ then f is nicely log-analytically prepared in x.*

Remark 3.12 *Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let E be a set of positive definable functions on C. Let $f : C \to \mathbb{R}$ be (e, r)–prepared in x with respect to E. Then there is a set $\mathcal{E}$ of C–heirs such that f can be constructed from $E \cup \mathcal{E}$.*

Proof We do an induction on e. For $e = -1$ there is nothing to show.
$e - 1 \to e$: Let $(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$ be a preparing tuple for f where

$b := (b_1, \ldots, b_s)$, $\exp(d) := (\exp(d_1), \ldots, \exp(d_s))$, and $\exp(c), \exp(d_1), \ldots, \exp(d_s)$ $\in E$. By the inductive hypothesis there is a set $\mathcal{E}'$ of C–heirs such that c and $d_1, \ldots, d_s$ can be constructed from $E \cup \mathcal{E}'$. By Remark 1.8 we see that $\exp(c), \exp(d_1), \ldots, \exp(d_s)$ can be constructed from $E \cup \mathcal{E}'$. Because a, $b_1, \ldots, b_s$ and $\Theta_0, \ldots, \Theta_r$ are C–nice there is a set $\bar{\mathcal{E}}$ of C–heirs such that a, $b_1, \ldots, b_s$ and $\Theta_0, \ldots, \Theta_r$ can be constructed from $\bar{\mathcal{E}}$. Set $\mathcal{E} := \bar{\mathcal{E}} \cup \mathcal{E}'$. Then $\eta := (a, \exp(c), b_1, \ldots, b_s, \exp(d_1), \ldots, \exp(d_s))$ can be constructed from $E \cup \mathcal{E}$. Note also that $y_0, \ldots, y_r$ can be constructed from $\mathcal{E}$ (compare with the proof of Remark 2.51). Let $k := 2 + 2s + r + 1$. With a similar argument as in the proof of (1) in Proposition 2.46 or Remark 3.7 we see that there is a globally subanalytic $F : \mathbb{R}^k \to \mathbb{R}$ such that $f(t,x) = F(\eta(t), y_0(t,x), \ldots, y_r(t,x))$ for every $(t,x) \in C$. With Remark 1.8 we are done. $\square$

Now we are ready to formulate and prove Theorem 0.5.

Theorem 3.13 *Let $e \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let E be a set of positive definable functions on X. Let $f : X \to \mathbb{R}$ be a function with exponential number at most e with respect to E. Then there is $r \in \mathbb{N}_0$ and a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is a finite set P of positive definable functions on C and $\Theta := (\Theta_0, \ldots, \Theta_r)$ such that the function $f|_C$ is (e,r)–prepared in x with center Θ with respect to P. Additionally the following hold.*

1. *For every $g \in \log(P)$ there is $m \in \{-1, \ldots, e-1\}$ such that g is (m,r)–prepared in x with center Θ with respect to P.*
2. *The following condition $(+_e)$ is satisfied: If $g \in \log(P)$ is (l,r)–prepared in x with center Θ with respect to P for $l \in \{-1, \ldots, e-1\}$ then g is a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$ restricted to C which have exponential number at most l with respect to E.*

Proof By 3.9 we may assume that f is e–prepared with respect to a finite set $L := \{g_1, \ldots, g_m\}$ of log-analytic functions and a finite set Q of positive definable functions with the following property:

Every $g \in \log(Q)$ is l–prepared with respect to L and Q for an $l \in \{-1, \ldots, e-1\}$. Additionally if $g \in \log(Q)$ is l–prepared for $l \in \{-1, \ldots, e-1\}$ with respect to L and Q then g is a finite $\mathbb{Q}$–linear combination of functions from $\log(E)$ which have exponential number at most l with respect to E. $(*_e)$

Let $r \in \mathbb{N}_0$ be such that $g_1, \ldots, g_m$ are log-analytic of order at most r. By Theorem 2.60 there is a definable cell decomposition $\mathcal{C}$ of $X_{\neq 0}$ such that $g_1|_C, \ldots, g_m|_C$ are nicely r–log-analytically prepared in x in a simultaneous way for every $C \in \mathcal{C}$. Fix $C \in \mathcal{C}$ and the corresponding center Θ for the r–logarithmic preparation of $g_1|_C, \ldots, g_m|_C$.

Claim 3.14 *Let $l \in \{-1, \ldots, e\}$ and $h \in \log(Q) \cup \{f\}$ be l–prepared with respect to L and Q. Then $h|_C$ is (l, r)–prepared in x with center Θ with respect to $P := Q|_C$.*

Proof We do an induction on l. If $l = -1$ it is clear. So assume $l \geq 0$. Since $h|_C$ is l–prepared with respect to $\{g_1|_C, \ldots, g_m|_C\}$ we have that $h|_C = \mu e^{\sigma} \tilde{v}(\nu_1 e^{\tau_1}, \ldots, \nu_k e^{\tau_k})$ where $k \in \mathbb{N}$, the functions $\mu, \nu_1, \ldots, \nu_k \in L|_C$ are nicely r–log-analytically prepared in x with center Θ, the functions $\sigma, \tau_1, \ldots, \tau_k \in \log(P)$ are $(l-1)$–prepared with respect to P and $L|_C$ and $\nu_j(t, x) e^{\tau_j(t,x)} \in [-1, 1]$ for every $(t, x) \in C$ and $j \in \{1, \ldots, k\}$. Additionally, $\tilde{v}$ is a power series which converges absolutely on an open neighborhood of $[-1, 1]^k$ with $\tilde{v}([-1, 1]^k) \subset \mathbb{R}_{>0}$. By the inductive hypothesis we see that σ and $\tau_1, \ldots, \tau_k$ are $(l-1, e)$–prepared in x with center Θ with respect to P.

With composition of power series we see that there is a C–nice $a : \pi(C) \to \mathbb{R}$, a nice r–logarithmic scale $\mathcal{Y}$ with center Θ, $s \in \mathbb{N}$, $q, p_1, \ldots, p_s \in \mathbb{Q}^{r+1}$, a tuple $b = (b_1, \ldots, b_s)$ of C–nice functions on $\pi(C)$, a tuple $\exp(d) := (\exp(d_1), \ldots, \exp(d_s))$ of definable functions on C with $d_j \in \{\tau_1, \ldots, \tau_k\} \cup \{0\}$ for every $j \in \{1, \ldots, s\}$ such that $h(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q} e^{\sigma(t,x)} v(\phi_1(t, x), \ldots, \phi_s(t, x))$ for every $(t, x) \in C$ where for $j \in \{1, \ldots y, s\}$,

$$\phi_j : C \to [-1, 1], \qquad (t, x) \mapsto b_j(t)|\mathcal{Y}(t, x)|^{\otimes p_j} \exp(d_j(t, x)),$$

v is a power series which converges on an open neighborhood of $[-1, 1]^s$, and $v([-1, 1]^s) \subset \mathbb{R}_{>0}$. So we see that h is (l, r)–prepared in x with respect to P. (By (1) in Remark 3.6 we have $e^0 \in Q$ since $l \geq 0$.) □

Because f is e–prepared with respect to Q we see by the claim that $f|_C$ is (e, r)–prepared in x with center Θ with respect to P. With the claim we obtain that $g|_C$ is (l, r)–prepared in x with center Θ with respect to P for every $g \in \log(Q)$ which is l–prepared with respect to Q. So with $(*_e)$ we see that $(+_e)$ is satisfied. □

Theorem C allows us to study specific classes of definable functions and discuss various analytic properties of them. One example are the so-called *restricted log-exp-analytic functions*. These are definable functions which are compositions of log-analytic functions and exponentials of locally bounded functions (see Definition 2.5 in Opris [12]). For a restricted log-exp-analytic function $f : X \to \mathbb{R}$ we find a pair $(e, r) \in (\mathbb{N}_0 \cup \{-1\}) \times \mathbb{N}_0$ and a decomposition $\mathcal{C}$ of X into finitely many definable cells such that for every $C \in \mathcal{C}$ there is a set P of locally bounded functions on C with respect to X such that $f|_C$ is (e, r)–prepared with respect to P. Consequences of this observation are the following. On the one hand differentiability properties like strong quasianalyticity or a parametric version of Tamm's theorem could be established for restricted log-exp-analytic functions (see [12]) and on the other hand it could be shown that a real analytic restricted log-exp-analytic function has a holomorphic extension which is again restricted log-exp-analytic (see Opris [11]). (These are generalizations of the corresponding results of van den Dries and Miller [4] and Kaiser [7] in the globally subanalytic setting respectively.) So restricted log-exp-analytic functions share their properties with globally subanalytic ones from the point of analysis, but not from the point of o–minimality.

I am hopeful that these results and proofs may serve as stepping stones towards a better understanding of the o–minimal structure $\mathbb{R}_{\text{an,exp}}$ by identifying more different interesting classes of definable functions and proving analytic properties of them.

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