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A computational aspect of the Lebesgue differentiation theorem

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Abstract: Given an L_1-computable function, f, we identify a canonical representative of the equivalence class of f, where f and g are equivalent if and only if $\int |f-g|$ is zero. Using this representative, we prove a modified version of the Lebesgue Differentiation Theorem. Our theorem is stated in terms of Martin-Löf random points in Euclidean space.

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1 Introduction

The Lebesgue Differentiation Theorem is a fundamental theorem in measure theory which generalizes the fundamental theorem of calculus. The Lebesgue Differentiation Theorem states that given $f \in L_1([0,1]^d)$,

$$f(x) = \lim_{Q \searrow x} \frac{\int_Q f}{\mu(Q)}$$

for almost every x where Q is a cube in $[0,1]^d$ containing x. In this paper, we look at the theorem in the context of computability theory. A proof of the Lebesgue Differentiation theorem can be found in the book of Wheeden-Zygmund [6] (p. 101-109) and with some work we can modify this proof for L_1-computable functions which are defined in Theorem 2.1. The final result we obtain will be a modified version of the Lebesgue Differentiation Theorem and will hold for all x which are Martin-Löf random. Due to the nature of Lebesgue integration, rather than working with actual functions f, it will be more useful to work with canonical representatives of f based on the equivalence relation

$$f \sim g \Leftrightarrow ||f-g||_1 = 0.$$

In this paper, we will prove such a canonical representative exists, and is well defined. Eventually, upon using some ideas from the original proof and creating some new

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tests for randomness, we will prove Theorem 5.1. A useful reference on computable analysis is given by Pour-El and Richards [4] and many of the ideas and terms in this paper come from that book.

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The referee has pointed out a result of Demuth [2] which says approximately the following: any computable function with bounded variation is differentiable at the Martin-Löf random reals. Due to notational and other difficulties, we have not been able to determine the exact content of Demuth's result and to what extent it may overlap with out results.

2 Some Notation and Definitions

The functions we consider in this paper will be real-valued and Lebesgue measurable with measure μ on the unit cube $[0,1]^d \subset \mathbb{R}^d$. Here d is a fixed positive integer. Our short description of this will be $f \in L_1([0,1]^d)$. Our norm will be the L_1-norm which is defined by

$$||f||_1 = \int_{[0,1]^d} |f| = \int_{x\in[0,1]^d} |f(x)|\, d\mu(x).$$

Note that L_1 functions have finite Lebesgue integral.

The following definition was provided by Pour-El and Richards [4].

Definition 2.1 *A function $f \in L_1([0,1]^d)$ is called L_1-computable if there exists a computable sequence of polynomials $f_n \in \mathbb{Q}[x]$ such that for all n*

$$||f - f_n||_1 < \frac{1}{2^n}$$

For future reference, note that we can easily find a computable sequence of rational numbers (D_n), depending on f_n, where D_n is an upper bound of the maximum gradient of each f_n, $\max_x\{|\nabla f_n(x)| : x \in [0,1]^d\}$

Definition 2.2 *Let f be an L_1 function and let Q denote a rational cube (that is, the coordinates of the vertices of Q are rational). Given x, we consider those Q containing*

x with edges parallel to the coordinate axes. Then, the indefinite integral of f is said to be differentiable at x if

$$\lim_{Q \searrow x} \frac{\int_Q f}{\mu(Q)}$$

exists.

Lemma 2.3 *Given a rational cube $Q \subseteq [0,1]^d$ and an L_1-computable function f, we can effectively find the computable real number $\int_Q f$.*

Proof To show that $\int_Q f$ exists and is computable we need to find a recursive sequence of rational numbers, (c_n) such that

$$\left| c_n - \int_Q f \right| < \frac{1}{2^n}$$

for all n. By Theorem 2.1, there exists a computable sequence of polynomials with rational coefficients, f_n, such that

$$||f - f_n||_1 \leq \frac{1}{2^n}.$$

We want to say that $c_n = \int_Q f_n$. First we show that $\int_Q f_n$ is rational for all n. To do this, recall that a polynomial is just a sum of monomials and since the integral of a sum is just the sum of integrals, we can just consider the integral of a monomial over Q. Note that in the following calculation $p_l \leq q_l$ for all l and $(q_1, \dots, q_d)$ and $(p_1, \dots, p_d)$ are diagonal vertices of Q.

$$\int_Q a \cdot x_1^{k1} \cdot x_2^{k2} \cdot \dots \cdot x_d^{kd} = \int_{p_d}^{q_d} \int_{p_{d-1}}^{q_{d-1}} \dots \int_{p_1}^{q_1} a \cdot x_1^{k1} \cdot x_2^{k2} \cdot \dots \cdot x_d^{kd}$$

$$= \frac{a(q_1^{k_1+1} - p_1^{k_1+1})(q_2^{k_2+1} - p_2^{k_2+1}) \dots (q_d^{k_d+1} - p_d^{k_d+1})}{(k_1+1)(k_2+1)\dots(k_d+1)}$$

This is a rational number because by the definition of f_n, a must be rational and $q_1, p_1, \dots, q_d, p_d$ are also rational because they are the coordinates of our rational cube Q. Thus, we see that $\int_Q f_n$ is rational for all n. Also, since (f_n) is a recursive sequence, $(\int_Q f_n)$ is also a recursive sequence. We say this because $\int_Q f_n$ can be re-written as the integral over Q of the elements of the sequence (f_n). Now that we have our recursive sequence of rational numbers, we just need to show that $(\int_Q f_n)$ converges to $\int_Q f$ at a nice rate.

$$\begin{aligned}\left|\int_Q f_n - \int_Q f\right| &\leq \int_Q |f_n - f| \\ &\leq \int_{[0,1]^d} |f_n - f| \\ &\leq \frac{1}{2^n}\end{aligned}$$

This proves that $\int_Q f$ exists and is computable. □

We now define Σ_1^0 sets.

Definition 2.4 *A set $S \subseteq \mathbb{N}^k \times [0,1]^d$ is called Σ_1^0 if there exists a recursive predicate $R \subseteq \mathbb{N}^{k+1} \times (\mathbb{Q} \cap [0,1])^{2d}$ such that*

$$\begin{aligned}S =& \{\langle m_1, \ldots, m_k, x_1, \ldots, x_d\rangle : (\exists j \in \mathbb{N})(\exists a_1, b_1, \ldots, a_d, b_d \in \mathbb{Q}) \\ & (R(j, m_1, \ldots, m_k, a_1, b_1, \ldots, a_d, b_d) \wedge a_1 < x_1 < b_1 \wedge \ldots \wedge a_d < x_d < b_d)\}.\end{aligned}$$

Definition 2.5 *A sequence of sets $(U_n) \subseteq [0,1]^d$ is called uniformly Σ_1^0 if the predicate $S \subseteq \mathbb{N} \times [0,1]^d$ where*

$$S(n, x) \equiv x \in U_n$$

is Σ_1^0

The next proposition provides a useful property of Σ_1^0 sets.

Proposition 2.6 *The class of Σ_1^0 sets is closed under the existential number quantifier.*

The next definition was first given by Martin-Löf [3].

Definition 2.7 *A point $x \in \mathbb{R}^d$ is Martin-Löf random if x does not lie in the intersection of any uniformly Σ_1^0 sequence (V_k) such that $\mu(V_k) \leq \frac{1}{2^k}$ for each k.*

Another characterization of random points in $\mathbb{R}^d$ is given by Solovay's Lemma, as proven by Simpson [5]. The proof given by Simpson [5] is given for sets in the Cantor space, but the proof applies here as well.

Lemma 2.8 (Solovay's Lemma) *Suppose that* (V_k) *is a sequence of uniformly* Σ_1^0 *sets in* $[0,1]^d$ *such that*

$$\sum_{k=1}^{\infty} \mu(V_k) < \infty.$$

Then for any random $x \in [0,1]^d$, x *lies in only finitely many* V_k.

Before we begin the proof of our modified Lebesgue Differentiation Theorem, we will need a few concepts and results to help set up the proof of the theorem.

3 A canonical representative of f

Lemma 3.1 (Chebyshev Inequality) *Given an* L_1*-computable function* f, *and* $\varepsilon > 0$, *let*

$$S(f,\varepsilon) = \{x : |f(x)| > \varepsilon\}.$$

Then

$$\mu(S(f,\varepsilon)) \leq \frac{||f||_1}{\varepsilon}.$$

Proof Consider $||f||_1$.

$$||f||_1 = \int_{[0,1]^d} |f| \geq \int_{S(f,\varepsilon)} |f| \geq \int_{S(f,\varepsilon)} \varepsilon \geq \varepsilon \cdot \mu(S(f,\varepsilon))$$

The result follows. □

The next lemma is based on Proposition 4.1 found in a paper by Brown, Guisto and Simpson [1].

Lemma 3.2 *Let* f *be* L_1*-computable. Then, there exists a uniformly* Σ_1^0 *sequence of sets* V_k, $k \in \mathbb{N}$ *such that* $\mu(V_k) \leq \frac{1}{2^{k-3}}$, *and for all* $x \notin V_k$ *and* $n \geq k$ *we have*

$$|f_i(x) - f_{2n}(x)| \leq \frac{1}{2^n}$$

for all $i \geq 2n$.

Proof Let f_n and D_n be as in Theorem 2.1. Let $V_k = \{x | (\exists n \geq k)(\exists i \geq 2n)(|f_i(x) - f_{2n}(x)| > \frac{1}{2^n})\}$. We want to show that the V_k are Σ^0_1. Since f_n is continuous for all n, $x \in V_k$ if and only if there exists a ball around x contained in V_k. For this reason, we can rewrite V_k as follows,

$$x \in V_k \equiv (\exists n \geq k)(\exists i \geq 2n)(\exists m \in \mathbb{N})(\exists a_1, b_1, \ldots, a_d, b_d \in \mathbb{Q}) \text{ such that}$$
$$(a_1 < x_1 < b_1, \ldots, a_d < x_d < b_d) \wedge \left(|f_i(a) - f_{2n}(a)| > \frac{1}{2^n} + \frac{1}{2^m} \right) \wedge$$
$$\left((D_i + D_{2n}) \cdot |a - b| < \frac{1}{2^m} \right)$$

Here n and i are natural numbers and $a = \langle a_1, \ldots, a_d \rangle$ and $b = \langle b_1, \ldots, b_d \rangle$. Define the predicate R by:

$$R(i, a_1, b_1, \ldots, a_d, b_d) = \left(|f_i(a) - f_{2n}(a)| > \frac{1}{2^n} + \frac{1}{2^m} \right) \wedge$$
$$\left((D_i + D_{2n}) \cdot |a - b| < \frac{1}{2^m} \right)$$

is a recursive predicate, so by Theorem 2.4 and Theorem 2.5 and Theorem 2.6, we can see that V_k is Σ^0_1 and the sequence (V_k) is uniformly Σ^0_1.

Now, we need to look at the measure of V_k. Note that

$$V_k \subseteq \bigcup_{n=k}^{\infty} S\left(\sum_{i=2n}^{\infty} |f_{i+1}(x) - f_i(x)|, \frac{1}{2^n} \right).$$

Using the previous lemma, we can conclude the proof as follows:

$$
\begin{aligned}
\mu(V_k) &\leq \mu\left(\bigcup_{n=k}^{\infty} S\left(\sum_{i=2n}^{\infty} |f_{i+1}(x) - f_i(x)|, \frac{1}{2^n}\right)\right) \\
&\leq \sum_{n=k}^{\infty} \mu\left(S\left(\sum_{i=2n}^{\infty} |f_{i+1}(x) - f_i(x)|, \frac{1}{2^n}\right)\right) \\
&= \sum_{n=k}^{\infty} 2^n \sum_{i=2n}^{\infty} ||f_{i+1} - f_i||_1 \\
&\leq \sum_{n=k}^{\infty} 2^n \sum_{i=2n}^{\infty} \frac{1}{2^{i-1}} \\
&\leq \sum_{n=k}^{\infty} \frac{2^n}{2^{2n-2}} \\
&= \sum_{n=k}^{\infty} \frac{1}{2^{n-2}} \\
&= \frac{1}{2^{k-3}}
\end{aligned}
$$

Thus we have our V_k and by the definition of V_k, for all x not in V_k and $n \geq k$,

$$|f_i(x) - f_{2n}(x)| \leq \frac{1}{2^n}$$

for all $i \geq n$ □

Lemma 3.3 *Let f be L_1-computable. Then $\lim_{n\to\infty} f_n(x)$ exists for all random x.*

Proof From the previous lemma, we can see that

$$\sum_{k=1}^{\infty} \mu(V_k) < \infty.$$

Since the sum is finite, we can use Solovay's Lemma and say that any random x will only be in finitely many V_k. So, for some large k and $\forall n \geq k$, we can see that

$$|f_i(x) - f_{2n}(x)| \leq \frac{1}{2^n}$$

for all $i \geq 2n$. From this we can see that f_n converges uniformly for $x \notin V_k$, and the limit exists. □

Definition 3.4 *Let $f \in L_1([0,1]^d)$. We define $\widehat{f}(x) : [0,1]^d \to \mathbb{R}$ to be*

$$\widehat{f}(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x \text{ is random} \\ 0 \text{ otherwise} \end{cases}$$

By Theorem 3.3 $\lim_{n\to\infty} f_n(x)$ exists for all random x so we know our new function is well-defined. We want to claim that $\widehat{f}$ is a canonical representation of the equivalence class of f ($f \sim g \Leftrightarrow ||f-g||_1 = 0$). The next two lemmas will prove that this is indeed the case.

Lemma 3.5 $\int_{[0,1]^d} \left| f - \widehat{f} \right| \, dx = 0$

Proof Let $E_\varepsilon = \left\{ x : \left| f(x) - \widehat{f}(x) \right| > \varepsilon \right\}$ and consider the set E_0. If this is a set of measure zero, then the result follows. Suppose, however, that this is not the case. Then, there exists some small $\varepsilon > 0$ such that $\mu\{E_\varepsilon\} > \varepsilon$. Now, by Theorem 3.2, for all random x, there exists k large such that $x \notin V_k$ and for all $n \geq k$, $|f_i(x) - f_{2n}(x)| \geq \frac{1}{2^n}$ for all $i \geq k$. By the definition of $\widehat{f}$ we can also say that $\left| \widehat{f}(x) - f_{2n}(x) \right| \geq \frac{1}{2^n}$. So for n such as the one above and $x \notin V_k$,

$$\begin{aligned} \mu\{x : |f(x) - f_{2n}(x)| > \varepsilon - \frac{1}{2^n}\} &> \mu\left\{ x : \left| \widehat{f}(x) - f(x) \right| - \left| \widehat{f}(x) - f_{2n}(x) \right| > \varepsilon - \frac{1}{2^n} \right\} \\ &> \mu\left\{ x : \left| \widehat{f}(x) - f(x) \right| - \frac{1}{2^n} > \varepsilon - \frac{1}{2^n} \right\} \\ &= \mu\left\{ x : \left| \widehat{f}(x) - f(x) \right| > \varepsilon \right\} \\ &> \varepsilon - \frac{1}{2^{n-3}} \end{aligned}$$

By Theorem 3.1,

$$||f - f_{2n}||_1 \geq \left(\varepsilon - \frac{1}{2^n} \right) \cdot \mu \left\{ x : |f(x) - f_{2n}(x)| > \varepsilon - \frac{1}{2^n} \right\} > \left(\varepsilon - \frac{1}{2^n} \right) \left(\varepsilon - \frac{1}{2^{n-3}} \right).$$

This is a contradiction because $||f - f_{2n}||_1 \to 0$ at $n \to \infty$. It follows that our assumption is incorrect and therefore, E_0 has measure zero. □

Lemma 3.6 *Given two L_1-computable functions, f, g*

$$\int_{[0,1]^d} |f - g| = 0 \;\; \textit{iff} \; \widehat{f} = \widehat{g}.$$

Proof ($\Leftarrow$) Suppose $\widehat{f} = \widehat{g}$. Then, $f(x) = g(x)$ for all x except for a set of measure zero. The result follows.

($\Rightarrow$) Suppose $\int_{[0,1]^d} |f - g| = 0$. Then,

$$\begin{aligned} ||f_n - g_n||_1 &= ||f_n - f + f - g + g - g_n||_1 \\ &\leq ||f_n - f||_1 + ||f - g||_1 + ||g - g_n||_1 \\ &\leq \frac{2}{2^n} \end{aligned}$$

This is a useful fact that will be used a little later. First we look at another consequence of our given assumption.

$$\begin{aligned} \int_{[0,1]^d} |f - g| &= \int_{[0,1]^d_{\text{random}}} |f - g| \\ &= \int_{[0,1]^d} \left|\widehat{f} - \widehat{g}\right| \end{aligned}$$

This means that $\widehat{f} = \widehat{g}$ except on a set of measure 0, E.

$$E = \left\{x \,|\, \widehat{f} \neq \widehat{g}\right\} = \left\{x | \lim_{n\to\infty} (f_n(x) - g_n(x)) \neq 0\right\}.$$

We would like to show that there cannot be any random $x \in E$. Let

$$V_n^k = \left\{x : |f_n(x) - g_n(x)| > \frac{1}{2^k}\right\}$$

The sets V_n^k will be our test for randomness. Using sets similar to the ones used in the proof of Theorem 3.2, we can show that for a fixed k the sequence (V_n^k) is uniformly Σ_1^0.

We would now like to use Solovay's Lemma. To do that, we need to show that the sum of the measures of V_n^k over all n is finite. To do this, we will use Theorem 3.1 again.

$$\begin{aligned} \mu\left(V_n^k\right) &= \mu\left(x : |f_n(x) - g_n(x)| > \frac{1}{2^k}\right) \\ &\leq 2^k \cdot ||f_n - g_n||_1 \\ &\leq \frac{2^k}{2^{n-1}} \end{aligned}$$

For a fixed k this is a geometric series, so $\sum_{n=1}^{\infty} \left|V_n^k\right| < \infty$. By Solovay's Lemma, for any fixed k, x can only be in finitely many V_n^k. Therefore for a random x, $\lim_{n\to\infty} |f_n(x) - g_n(x)| \leq \frac{1}{2^k}$ for all k meaning that $\lim_{n\to\infty} |f_n(x) - g_n(x)| = 0$. This shows that there cannot be a random x in E. By the definition of $\widehat{f}$ and $\widehat{g}$, there cannot be a non-random x in E either. This means that E is empty, and $\widehat{f} = \widehat{g}$ □

From the last two lemmas, we can see that $\widehat{f}$ is a canonical representative of the equivalence class of f. The next section provides some results necessary for the proof of the main theorem.

4 Some Important Lemmas

First, will will prove the Lebesgue Differentiation theorem for continuous functions.

Proposition 4.1 *Let $f \in L_1\left(\mathbb{R}^d\right)$ be a continuous function. Then, the indefinite integral of f is differentiable and its derivative is equal to $f(x)$ for all $x \in \mathbb{R}^d$.*

Proof The proof is clear from the following calculations. If f is continuous at x and Q is a rational cube containing x, then

$$\begin{aligned}\left|\frac{1}{\mu(Q)}\int_Q f(y)\,dy - f(x)\right| &= \left|\frac{1}{\mu(Q)}\int_Q [f(y)-f(x)]\,dy\right| \\ &\leq \frac{1}{\mu(Q)}\int_Q |f(y)-f(x)|\,dy \\ &\leq \sup_{y\in Q}|f(y)-f(x)|,\end{aligned}$$

which tends to zero as Q shrinks to x. □

Using this fact we develop the idea for the proof. By Theorem 2.1, we can approximate our function f using continuous polynomials. Using this, we can approximate the indefinite integral of f and create a test for randomness. To do all this, we will need a few lemmas.

Lemma 4.2 (Simple Vitali Lemma) *Let E be a subset of $[0,1]^d$, and let K be a collection of cubes Q in $[0,1]^d$ covering E. Then there exists a positive constant β, depending only on d, and a finite number of disjoint cubes $Q_1,\ldots,Q_N$ in K such that*

$$\sum_{j=1}^{N}\mu(Q_j) \geq \beta\cdot\mu(E)$$

A proof of the Simple Vitali Lemma is given by Wheedon and Zygmund [6] on page 102.

Definition 4.3 *Consider a function* $g : [0,1]^d \to \mathbb{R}$ *that is integrable over every cube* $Q \subseteq [0,1]^d$. *Let*

$$g^*(x) = \sup\left\{\frac{1}{\mu(Q)}\int_Q |g(y)|dy\right\},$$

where the supremum is taken over every cube Q *with center* x *and edges parallel to the coordinate axes. This function,* g^* *is called the Hardy–Littlewood maximal function of* g.

This is a slightly modified version of the definition of the Hardy–Littlewood maximal function.

Definition 4.4 *Consider an* L_1 *function* $g : [0,1]^d \to \mathbb{R}$. *We say* g *belongs to weak* $L([0,1]^d)$ if there exists c independent of α such that

$$\mu\left\{x \in [0,1]^d : |g(x)| > \alpha\right\} \leq \frac{c}{\alpha} \quad (\alpha > 0).$$

Lemma 4.5 (Hardy–Littlewood) *Given an* L_1 *function* $f : [0,1]^d \to \mathbb{R}$, f^* *belongs to weak* $L([0,1]^d)$. *Moreover, there is a constant* c *independent of* f *and* α *such that*

$$\mu\left\{x \in [0,1]^d : f^*(x) > \alpha\right\} \leq \frac{c}{\alpha}\int_{[0,1]^d} |f|, \alpha > 0$$

Proof Since the domain of f and f^* is $[0,1]^d$ if we fix $\alpha > 0$ we can say that the measure of the set

$$E = \{x : f^*(x) > \alpha\}$$

is finite. If $x \in E$, there is a rational cube Q_x containing x such that $\frac{1}{\mu(Q_x)}\int_{Q_x} |f| > \alpha$, or

$$\mu(Q_x) < \frac{1}{\alpha}\int_{Q_x} |f|.$$

The collection of such Q_x covers E, so by Theorem 4.2, there exist $\beta > 0$ and $x_1, \ldots x_N \in E$ such that $Q_{x_1}, \ldots, Q_{x_N}$ are disjoint and $\mu(E) < \frac{1}{\beta}\sum_{j=1}^N \mu(Q_{x_j})$. Putting everything together, we get

$$\mu(E) < \frac{1}{\beta}\sum_{j=1}^N \frac{1}{\alpha}\int_{Q_{x_j}} |f| = \frac{1}{\beta\alpha}\int_{\bigcup_{j=1}^N Q_{x_j}} |f| \leq \frac{1}{\beta\alpha}\int_{[0,1]^d} |f|.$$

This proves the Hardy–Littlewood Lemma. □

Definition 4.6 *Given $f \in L_1$ and $\varepsilon > 0$, let $S^*(f,\varepsilon)$ be the union of all Q such that*

$$\frac{\int_Q |f|}{\mu(Q)} > \varepsilon.$$

Note that according to this definition, and the Hardy–Littlewood Lemma,

$$\mu(S^*(f,\varepsilon)) \leq \frac{c||f||_1}{\varepsilon}.$$

Lemma 4.7 *Let c be the constant from the Hardy–Littlewood Lemma and let f be L_1-computable. Then, we can find sets V_k^* which are uniformly Σ_1^0 such that $\mu(V_k^*) \leq \frac{c}{2^{k-1}}$ and for all $x \notin V_k^*$ and $n \geq k$ we have that*

$$\frac{\int_Q |f - f_{2n}|}{\mu(Q)} \leq \frac{1}{2^n}$$

for all Q containing x.

Proof Let

$$V_k^* = \bigcup_{n=k}^{\infty} S^*\left(f - f_{2n}, \frac{1}{2^n}\right)$$

Since the union of countable many uniformly Σ_1^0 sets is Σ_1^0, we need to show that the sequence $\left(S^*\left(f - f_{2n}, \frac{1}{2^n}\right)\right)$ is uniformly Σ_1^0. Let

$$R_n(\varepsilon, a_1, b_1, \ldots, a_d, b_d) = \frac{\int_{a_d}^{b_d} \cdots \int_{a_1}^{b_1} f(x)}{(a_1 - b_1) \cdot \ldots \cdot (a_d - b_d)} > \varepsilon.$$

Then, we can write $S^*\left(f - f_{2n}, \frac{1}{2^n}\right)$ in the following way:

$$S^*\left(f - f_{2n}, \frac{1}{2^n}\right) = \Big\{x : \exists a_1, b_1, \ldots, a_d, b_d \in \mathbb{Q}_{[0,1]} | \left(R\left(\frac{1}{2^n}, a_1, b_1, \ldots, a_d, b_d\right)\right) \text{ and } a_1 < x_1 < b_1, \ldots, a_d < x_d < b_d\Big\}$$

This is a sequence of uniformly Σ_1^0 sets by Theorem 2.5 which means that each V_k^* is Σ_1^0 and the sequence (V_k^*) is uniformly Σ_1^0. Now we need to look at the measure of

V_k^*. We will use the Hardy–Littlewood Lemma.

$$\begin{aligned}\mu\left(V_k^*\right) &\leq \sum_{n=k}^{\infty}\mu\left(S^*\left(f - f_{2n}, \frac{1}{2^n}\right)\right)\\ &\leq \sum_{n=k}^{\infty} c\cdot 2^n \cdot ||f - f_{2n}||_1\\ &\leq \sum_{n=k}^{\infty}\frac{c2^n}{2^{2n}}\\ &\leq \sum_{n=k}^{\infty}\frac{c}{2^n}\\ &= \frac{c}{2^{k-1}}\end{aligned}$$

The result follows. □

There is one last lemma we need before we can prove the main result.

Lemma 4.8 *Let f be L_1-computable and let D_n and f_n be as in Theorem 2.1. Then for all k, $n \geq k$ and all $x \notin V_k \cup V_k^*$,*

$$\left|\widehat{f}(x) - \frac{\int_Q f}{\mu(Q)}\right| \leq \frac{1}{2^{n-1}} + D_{2n}\cdot \text{ (diameter of Q)}$$

for all rational cubes Q containing x. Here the sequence of V_k is from Theorem 3.2 and the sequence of V_k^* is from Theorem 4.7.

Proof Since D_n is an upper bound of the maximum gradient of each f_n, $\max\{|\nabla f_n| : x \in [0,1]^d\}$, we can use the Mean Value Theorem to say,

$$\left|f_{2n}(x) - \frac{\int_Q f_{2n}}{\mu(Q)}\right| \leq D_{2n}\cdot \text{ (diameter of Q)}$$

for all rational Q containing x. By Theorem 3.2 we have that

$$\left|\widehat{f}(x) - f_{2n}(x)\right| \leq \frac{1}{2^n}$$

and by Theorem 4.7,

$$\frac{\int_Q |f - f_{2n}|}{\mu(Q)} \leq \frac{1}{2^n}$$

for all $n \geq k$ and $x \notin V_k \cup V_k^*$. Combining these two, we get that

$$\begin{aligned}
\left|\widehat{f}(x) - \frac{\int_Q f}{\mu(Q)}\right| &= \left|\widehat{f}(x) - f_{2n}(x) + f_{2n}(x) - \frac{\int_Q f_{2n}}{\mu(Q)} + \frac{\int_Q f_{2n}}{\mu(Q)} - \frac{\int_Q f}{\mu(Q)}\right| \\
&\leq \left|\widehat{f}(x) - f_{2n}(x)\right| + \left|\frac{\int_Q f_{2n}}{\mu(Q)} - \frac{\int_Q f}{\mu(Q)}\right| + \left|f_{2n}(x) - \frac{\int_Q f_{2n}}{\mu(Q)}\right| \\
&\leq \frac{1}{2^n} + \frac{1}{2^n} + D_{2n} \cdot \text{ (diameter of Q)} \\
&\leq \frac{1}{2^{n-1}} + D_{2n} \cdot \text{ (diameter of Q)}
\end{aligned}$$

□

5 Main Result

Theorem 5.1 *Let f be an L_1-computable function. Let $\widehat{f}$ be the canonical representation of f as defined in Theorem 3.4. Then for all random x,*

$$\widehat{f}(x) = \lim_{Q \searrow x} \frac{\int_Q f}{\mu(Q)}$$

for Q containing x.

Proof Let f_n and D_n be as in Theorem 2.1. V_k from Theorem 3.2 and V_k^* from Theorem 4.7 form Martin-Löf tests. So, for a random x, there exists a large k such that $x \notin V_k \cup V_k^*$. We want to show that for all $\varepsilon > 0$ $\exists \delta > 0$ such that

$$\left|\widehat{f}(x) - \frac{\int_Q |f|}{\mu(Q)}\right| < \varepsilon$$

whenever the diameter of Q is less than δ. Choose n large so that $\frac{1}{2^{n-1}} < \frac{\varepsilon}{2}$ and let $\delta = \frac{\varepsilon}{2D_{2n}}$. Then, when the diameter of Q is less than δ,

$$\left|\widehat{f}(x) - \frac{\int_Q |f|}{\mu(Q)}\right| < \frac{1}{2^{n-1}} + \frac{\varepsilon}{2} < \varepsilon.$$

□

6 Closing Remarks

By only dealing with L_1-computable functions, our theorem seems at first to be less general than the original Lebesgue Differentiation Theorem. However, if we consider relativization, it can be seen that the statement proved in this paper is stronger. Any L_1 function is computable relative to some oracle and using this we can prove a relativized version of Theorem 5.1 pertaining to any function and provide a very specific set of measure zero, outside of which the Lebesgue Differentiation Theorem always holds.

In Theorem 5.1 we have proved that the Lebesgue Differentiation Theorem holds at x provided that x is a Martin-Löf random point in a Euclidean space. The natural question arises, is the converse true? That is, if we have that the Lebesgue Differentiation Theorem holds at x for all L_1-computable functions, is x necessarily random? This is an important question as the converse holding would give an alternative characterization of random points in Euclidean space.

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